Section 11.8: The Power Series

A **power series** is a series of the form

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

More generally, a series of the form

$$f(x-a) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called a **power series centered at *a*** or a **power series about *a***.

The domain of a power series is the set of all x for which the series converges.

Recall the **Ratio Test**:

The Ratio Test

(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (⇨ convergent).

(ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive.

Example 4. For what x-values does the power series $\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$ converge?

SOLUTION Here $a_n = x^n/(2n)!$ and, as $n \to \infty$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{[2(n+1)]!} \cdot \frac{(2n)!}{x^n}\right| = \frac{(2n)!}{(2n+2)!}|x|$$

$$= \frac{(2n)!}{(2n)!(2n+1)(2n+2)}|x| = \frac{|x|}{(2n+1)(2n+2)} \to 0 < 1$$

for all x. Thus, by the Ratio Test, the given series converges for all values of x.

Theorem For a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:

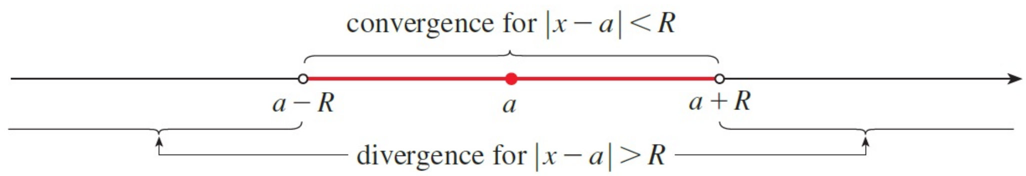
(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

The number R in case (iii) is called the **radius of convergence** of the power series. By convention, the radius of convergence is $R = 0$ in case (i) and $R = \infty$ in case (ii).

The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges.



We summarize here the radius and interval of convergence for each of the examples already considered in this section.

	Series	Radius of convergence	Interval of convergence
Example 1	$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
Example 2	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$
Example 3	$\sum_{n=0}^{\infty} n!\, x^n$	$R = 0$	$\{0\}$
Example 4	$\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$	$R = \infty$	$(-\infty, \infty)$

Example 6. Find the radius and the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$

SOLUTION If $a_n = n(x+2)^n/3^{n+1}$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n}\right| = \left(1+\frac{1}{n}\right)\frac{|x+2|}{3} \to \frac{|x+2|}{3} \quad \text{as } n \to \infty$$

Using the Ratio Test, we see that the series converges if $|x+2|/3 < 1$ and it diverges if $|x+2|/3 > 1$. So it converges if $|x+2| < 3$ and diverges if $|x+2| > 3$. Thus the radius of convergence is $R = 3$.

The inequality $|x+2| < 3$ can be written as $-5 < x < 1$, so we test the series at the endpoints -5 and 1. When $x = -5$, the series is

$$\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \tfrac{1}{3} \sum_{n=0}^{\infty} (-1)^n n$$

which diverges by the Test for Divergence [$(-1)^n n$ doesn't converge to 0]. When $x = 1$, the series is

$$\sum_{n=0}^{\infty} \frac{n(3)^n}{3^{n+1}} = \tfrac{1}{3} \sum_{n=0}^{\infty} n$$

which also diverges by the Test for Divergence. Thus the series converges only when $-5 < x < 1$, so the interval of convergence is $(-5, 1)$.