Section 2.5: Continuity

Problem 1. Evaluate the limit.

$$\lim_{x\to 0} e^{\frac{\sqrt{1-x}-1}{x}}$$

HINT: Use the theorem below that we learned in class.

Theorem 2. If f is continuous at b and $\lim_{x\to a} g(x) = b$ then

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b)\ .$$

Let function in the limit is the composition of the functions $f(x) = e^x$ and $g(x) = \frac{\sqrt{1-x}-1}{x}$. That is, $e^{\frac{\sqrt{1-x}-1}{x}} = f(g(x))$. Let us determine $\lim_{x\to 0} g(x)$. We have

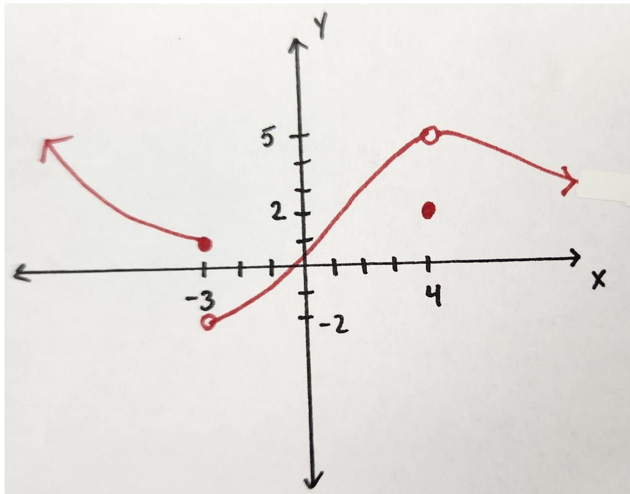
$$\begin{aligned}
\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{\sqrt{1-x}-1}{x} &= \lim_{x\to 0} \frac{\sqrt{1-x}-1}{x} \cdot \frac{\sqrt{1-x}+1}{\sqrt{1-x}+1} \\
&= \lim_{x\to 0} \frac{(1-x)-1}{x(\sqrt{1-x}+1)} \\
&= \lim_{x\to 0} \frac{-x}{x(\sqrt{1-x}+1)} \\
&= \lim_{x\to 0} \frac{-1}{\sqrt{1-x}+1} \\
&= \frac{-1}{\sqrt{1-0}+1} \\
&= -\frac{1}{2}
\end{aligned}$$

Since $f(x) = e^x$ is continuous on its domain, $\mathbb{R} = (-\infty, \infty)$, then by Theorem 2,

$$\lim_{x\to 0} e^{\frac{\sqrt{1-x}-1}{x}} = e^{\lim_{x\to 0} \frac{\sqrt{1-x}-1}{x}} = e^{-1/2}.$$

Problem 2. Sketch the graph of a function that satisfies the following

Jump discontinuity at -3,
Removable discontinuity at 4,
Continuous from the left at -3,
Value of the function at $x = 4$ is 2,
Continuous everywhere except at -3 and 4



Problem 3. Let

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x \leq 0 \\ 1 & \text{if } 0 < x < 2 \\ \dfrac{x^2 - 9}{x - 3} & \text{if } x \geq 2. \end{cases}$$

(a) Find the discontinuities of f and state the type of discontinuity.

(b) Determine whether f is continuous from the left, continuous from the right, or neither at each of the discontinuities you stated in part (a).

(a) We will determine continuity at the following places: $(-\infty, 0)$, $x = 0$, $(0, 2)$, $x = 2$, $(2, \infty)$.

On the interval $(-\infty, 0)$, the function $f(x) = x^2 + 1$, which is a polynomial. Since polynomials are continuous on $\mathbb{R}$, f is continuous on $(-\infty, 0)$.

Let us determine the continuity of f at 0. We have $f(0) = 0^2 + 1 = 1$. Checking left and right limits, we have

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 + 1 = 0^2 + 1 = 1,$$

and

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 = 1.$$

Therefore, $\lim_{x \to 0} f(x) = 1 = f(0)$, so f is continuous at 1.

On the interval $(0,2)$, the function $f(x) = 1$, which is a polynomial (constant functions are polynomials). Since polynomials are continuous on $\mathbb{R}$, f is continuous on $(0,2)$.

Let us determine the continuity of f at 2. We have $f(2) = \dfrac{2^2-9}{2-3} = 5$. Checking left and right limits, we have

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 1 = 1,$$

and

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x^2-9}{x-3} = \lim_{x\to 2^+} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 2^+} x+3 = 2+3 = 5.$$

Since the left and right limits are not equal, **f has a jump discontinuity at 2**.

On the interval $(2,\infty)$, the function $f(x) = \dfrac{x^2-9}{x-3}$, which is continuous on its domain, $\{x \mid x \neq 3\}$. Since 3 is in the interval $(2,\infty)$, f has a discontinuity at 3. Since

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3} x+3 = 3+3 = 6,$$

then **f has a removable discontinuity at 3**.

(b) In part (a), we showed that

$$\lim_{x\to 2^+} f(x) = 5 = f(2).$$

Therefore, **f is continuous from the right at 2**.

Since $f(3)$ does not exist, **f is neither continuous from the right or the left at 3**.

Problem 4. Use the Intermediate Value Theorem (IVT) to show that the equation $\ln(x) = x - \sqrt{x}$ has a solution within the interval $(2,3)$.

HINT: Let $f(x) = \ln(x) - x + \sqrt{x}$ and determine the signs of $f(2)$ and $f(3)$. Apply the same approach as we did in class for a similar problem. Please justify your steps in order to apply the IVT.

Intermediate Value Theorem. (**IVT**) Suppose that f is continuous on the closed interval $[a,b]$ and let N be any number between $f(a)$ and $f(b)$. Then there exists a number c in (a,b) such that $f(c) = N$.

Using the given hint, we let $f(x) = \ln(x) - x + \sqrt{x}$. Notice that the function is the sum of three functions, each of which is continuous on their domain. Since $\text{dom}(f) = (0,\infty)$, then f is continuous on $(0,\infty)$, and therefore in continuous on $(2,3)$. Since

$$f(2) = \ln(2) - 2 + \sqrt{2} \approx 0.107 > 0 \quad \text{and} \quad f(3) = \ln(3) - 3 + \sqrt{3} \approx -0.169 < 0,$$

then 0 is an intermediate value between $f(2)$ and $f(3)$. Since f is continuous on $(2,3)$, by the IVT, we know that there is an x-value c in $(2,3)$ such that $f(c) = 0$. That is, there is a value c, such that $\ln(c) - c + \sqrt{c} = 0$, which is equivalent to stating that the equation $\ln(x) = x - \sqrt{x}$ has a solution within the interval $(2,3)$.