MAT 1500 (Dr. Fuentes) Worksheet 3 SOLUTIONS - Sections 2.3 & 2.5

Section 2.3: Calculating Limits Using Limit Laws

Problem 1. Evaluate each of the following limits if they exist.

$(a)\ \lim_{x\to -3} \frac{x^2+3x}{x^2-x-12}$, $(b)\ \lim_{x\to 3} \frac{\frac{1}{x}-\frac{1}{3}}{x-3}$, $(c)\ \lim_{x\to 9} \frac{9-x}{3-\sqrt{x}}$ $(d)\ \lim_{x\to 4} |x-4|+2x$.

(a) $$\lim_{x\to -3} \frac{x^2+3x}{x^2-x-12} = \lim_{x\to -3} \frac{x(x+3)}{(x-4)(x+3)} = \lim_{x\to -3} \frac{x}{x-4} = \frac{-3}{(-3-4)} = \frac{3}{7}.$$

(b) $$\lim_{x\to 3} \frac{\frac{1}{x}-\frac{1}{3}}{x-3} = \lim_{x\to 3} \frac{\frac{1}{x}-\frac{1}{3}}{x-3} \cdot \frac{3x}{3x} = \lim_{x\to 3} \frac{3-x}{(x-3)3x} = \lim_{x\to 3} \frac{-(x-3)}{(x-3)3x} = \lim_{x\to 3} \frac{-1}{3x} = \frac{-1}{3\cdot 3} = \frac{-1}{9}$$

(c) $$\lim_{x\to 9} \frac{9-x}{3-\sqrt{x}} = \lim_{x\to 9} \frac{9-x}{3-\sqrt{x}} \cdot \frac{3+\sqrt{x}}{3+\sqrt{x}} = \lim_{x\to 9} \frac{(9-x)(3+\sqrt{x})}{9-x}$$
$$= \lim_{x\to 9} 3+\sqrt{x} = \lim_{x\to 9} 3 + \lim_{x\to 9} \sqrt{x} = 3 + \sqrt{\lim_{x\to 9} x}$$
$$= 3+\sqrt{9} = 6.$$

(d) We can express

$$|x-4|+2x = \begin{cases} x-4+2x & \text{if } x-4\geq 0 \\ -x+4+2x & \text{if } x-4<0 \end{cases} = \begin{cases} 3x-4 & \text{if } x\geq 4 \\ x+4 & \text{if } x<4. \end{cases}$$

Then since
$$\lim_{x\to 4^+} (|x-4|+2x) = \lim_{x\to 4} 3x-4 = 3\cdot 4-4 = 8$$
and
$$\lim_{x\to 4^-} (|x-4|+2x) = \lim_{x\to 4} x+4 = 4+4 = 8$$
are equal, then we must have
$$\lim_{x\to 4} (|x-4|+2x) = 8.$$

Section 2.5: Continuity

Problem 2. Use the Squeeze Theorem to show that $\lim_{x\to 0} \sqrt{x^4+x^2}\sin\left(\frac{\pi}{x}\right) = 0.$

Since π/x is a real number for any real value x, then we have

$$-1 \le \sin\left(\frac{\pi}{x}\right) \le 1.$$

Since $x^4 \ge 0$ and $x^2 \ge 0$, then certainly, $\sqrt{x^4+x^2} \ge 0$. Therefore, the following inequality holds:

$$-\sqrt{x^4+x^2} \le \sqrt{x^4+x^2}\sin\left(\frac{\pi}{x}\right) \le \sqrt{x^4+x^2}.$$

Since

$$\lim_{x\to 0}\sqrt{x^4+x^2} = \sqrt{\lim_{x\to 0}(x^4+x^2)} = \sqrt{0^4+0^2} = \sqrt{0} = 0,$$

and

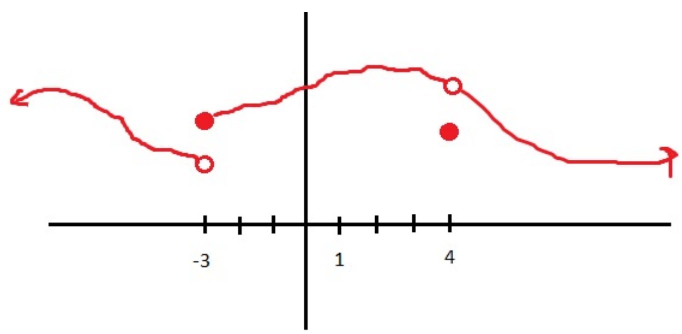
$$\lim_{x\to 0} -\sqrt{x^4+x^2} = -\lim_{x\to 0}\sqrt{x^4+x^2} = 0,$$

then by the Squeeze Theorem, we must have

$$\lim_{x\to 0}\sqrt{x^4+x^2}\sin\left(\frac{\pi}{x}\right) = 0.$$

Problem 3. Sketch the graph of a function f that is defined on the set of real numbers (meaning $\mathbb{R} = (-\infty, \infty)$) that is continuous, except for the following discontinuities:

Jump discontinuity at $x = -3$, removable discontinuity at $x = 4$.



Problem 4. Clearly explain why the function $f(x) = \frac{x^2}{\sqrt{x^4+2}}$ is continuous at every number in its domain. State the domain of the function.

The function f consists of a quotient of the two functions. The numerator is $g(x) = x^2$, a polynomial, and the denominator is $h(x) = \sqrt{x^4+2}$, a root function. Since both polynomials and root functions are continuous everywhere on their domains, then f is also continuous everywhere on its domain.

The domain of f is

$$\text{domain}(f) = \left\{x \mid x^4+2 \ge 0\right\} = (-\infty, \infty).$$

Notice that all real numbers satisfy $x^4+2 \ge 0$.