MAT 1500 (Dr. Fuentes) **Worksheet 2 (PART 2) - Sections 2.2 & 2.3**

Section 2.2: The Limit of a Function

Problem 1. Determine the limits below.

(a) $\lim_{x \to 1^+} \ln(\sqrt{x} - 1)$ (b) $\lim_{x \to 0^+} \ln(\sin(x))$

HINT: Remember, $f(x) = \ln(x)$ has a vertical asymptote at $x = 0$, since as $x \to 0^+$, $\ln(x) \to -\infty$.

(a) As $x \to 1^+$, $\sqrt{x} - 1 \to 0^+$. Then $\ln(\sqrt{x} - 1) \to -\infty$. That is, $\lim_{x \to 1^+} \ln(\sqrt{x} - 1) = -\infty$

(b) As $x \to 0^+$, $\sin(x) \to 0^+$. Then $\ln(\sqrt{x} - 1) \to -\infty$. That is, $\lim_{x \to 0^+} \ln(\sin(x)) = -\infty$.

Problem 2. Find the vertical asymptotes of the function below. Explain the behavior of the function on either side of the vertical aymptote (e.g., if $x = a$ is a v.a., explain whether the function goes to ∞ or $-\infty$ as $x \to a$.) Verify your answers by plotting the function in Maple.

$$f(x) = \frac{x^2 + 1}{3x - 2x^2}$$

Since

$$\frac{x^2 + 1}{3x - 2x^2} = \frac{x^2 + 1}{x(3 - 2x)},$$

we see that the denominator $x(3 - 2x) = 0$ when $x = 0$ and when $x = 3/2$. Note that

$$f(0) = \frac{0^2 + 1}{0(3 - 2(0))} = \frac{1}{0} \quad \text{and} \quad f(3/2) = \frac{(3/2)^2 + 1}{(3/2)(3 - 2(3/2))} = \frac{13/4}{0},$$

which idicates that both $x = 0$ and $x = 3/2$ are vertical asymptotes of f.

Let us determine the behavior of the function on either side of these asymptotes. As $x \to 0^+$, $x(3 - 2x) \to 0^-$, since $x > 0$ and $3 - 2x > 0$ by taking x sufficiently close to 0. Since the numerator $x^2 + 1 \to 1$, then

$$\lim_{x \to 0^+} \frac{x^2 + 1}{3x - 2x^2} = \infty.$$

As $x \to 0^-$, $x(3 - 2x) \to 0^+$, since $x < 0$ and $3 - 2x > 0$. Since the numerator $x^2 + 1 \to 1$, then

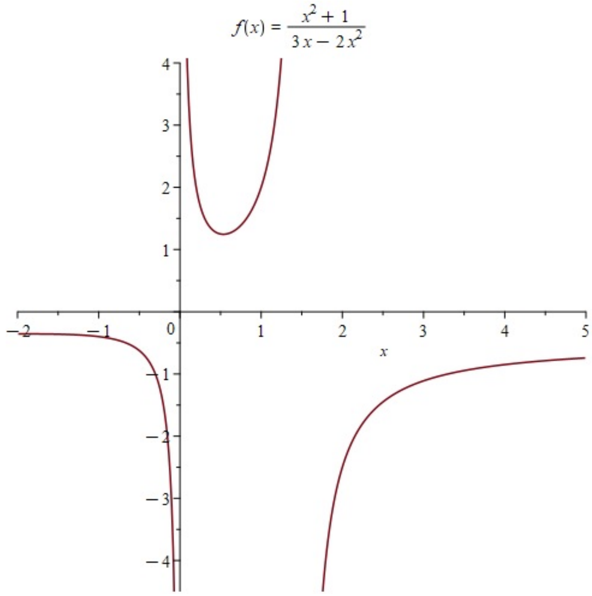
$$\lim_{x \to 0^-} \frac{x^2 + 1}{3x - 2x^2} = -\infty.$$

As $x \to (3/2)^+$, $x(3 - 2x) \to 0^-$, since $x > 0$ and $3 - 2x < 0$. Since the numerator $x^2 + 1 \to 13/4$, then

$$\lim_{x \to (3/2)^+} \frac{x^2 + 1}{3x - 2x^2} = -\infty.$$

As $x \to (3/2)^-$, $x(3 - 2x) \to 0^+$, since $x > 0$ and $3 - 2x < 0$, by taking x sufficiently close to 3/2. Since the numerator $x^2 + 1 \to 13/4$, then

$$\lim_{x \to (3/2)^-} \frac{x^2 + 1}{3x - 2x^2} = \infty.$$



Section 2.3: Calculating Limits Using Limit Laws

Problem 3. Evaluate each of the following limits if they exist.

(a) $\lim_{h\to 0} \frac{(h-2)^{-1}+2^{-1}}{h}$, (b) $\lim_{t\to 0} \frac{1}{t\sqrt{1+t}} - \frac{1}{t}$ (c) $\lim_{x\to -2} \frac{2-|x|}{2+x}$.

HINTS: (a) Express each term in the numerator as a fraction and then combine them into a single fraction by finding their least common denominator.
(b) Combine the fractions into a single fraction, then rationalize the numerator.
(c) When x is very close to -2, x is negative.

(a)

$$\lim_{h\to 0} \frac{(\,h-2\,)^{-1}+2^{-1}}{h} = \lim_{h\to 0} \frac{\frac{1}{h-2}+\frac{1}{2}}{h} = \lim_{h\to 0} \frac{\frac{2+(h-2)}{2(h-2)}}{h} = \lim_{h\to 0} \frac{2+(h-2)}{2h(h-2)}$$

$$= \lim_{h\to 0} \frac{h}{2h(h-2)} = \lim_{h\to 0} \frac{1}{2(h-2)} = \frac{1}{2(0-2)} = -\frac{1}{4}$$

(b)

$$\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t}\right) = \lim_{t\to 0} \frac{1-\sqrt{1+t}}{t\sqrt{1+t}} = \lim_{t\to 0} \frac{(1-\sqrt{1+t}\,)(1+\sqrt{1+t}\,)}{t\sqrt{t+1}\,(1+\sqrt{1+t}\,)} = \lim_{t\to 0} \frac{-t}{t\sqrt{1+t}\,(1+\sqrt{1+t}\,)}$$

$$= \lim_{t\to 0} \frac{-1}{\sqrt{1+t}\,(1+\sqrt{1+t}\,)} = \frac{-1}{\sqrt{1+0}\,(1+\sqrt{1+0}\,)} = -\frac{1}{2}$$

(c) Since x is approaching -2, we can assume that x < 0 by assuming x is very close to -2.
Since $|x| = -x$ for $x < 0$, we have

$$\lim_{x\to -2} \frac{2-|x|}{2+x} = \lim_{x\to -2} \frac{2-(-x)}{2+x} = \lim_{x\to -2} \frac{2+x}{2+x} = \lim_{x\to -2} 1 = 1.$$

Problem 4. Use the Squeeze Theorem to show that $\lim_{x \to 0^+} \sqrt{x}\, e^{\sin(\pi/x)} = 0$.

For any x-value, we know that

$$-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1.$$

Then

$$e^{-1} \leq e^{\sin(\pi/x)} \leq e^1.$$

Since $\sqrt{x} \geq 0$ (the square root of a number is always nonnegative), then

$$\sqrt{x}e^{-1} \leq \sqrt{x}e^{\sin(\pi/x)} \leq \sqrt{x}e.$$

Notice that

$$\lim_{x \to 0^+} \sqrt{x}e^{-1} = e^{-1} \lim_{x \to 0^+} \sqrt{x} = e^{-1}(0) = 0,$$

and similarly,

$$\lim_{x \to 0^+} \sqrt{x}e = 0.$$

Then by the Squeeze Theorem, we also have

$$\lim_{x \to 0^+} \sqrt{x}\, e^{\sin(\pi/x)} = 0.$$