The Product Rule can be applied the the product of **three of more functions** as well. For example, if $F(x) = f(x) \cdot g(x) \cdot h(x) = f(x)\,(g(x) \cdot h(x))$ then by the Product Rule,

$$\begin{aligned} F'(x) &= \left(\frac{d}{dx}f(x)\right) g(x)\cdot h(x) + f(x)\left(\frac{d}{dx}(g(x)\cdot h(x))\right) \\ &= \left(\frac{d}{dx}f(x)\right) g(x)\cdot h(x) + f(x)\left(\left(\frac{d}{dx}g(x)\right) h(x) + g(x)\left(\frac{d}{dx}h(x)\right)\right) \\ &= \left(\frac{d}{dx}f(x)\right) g(x)\cdot h(x) + f(x)\left(\frac{d}{dx}g(x)\right) h(x) + f(x)\cdot g(x)\left(\frac{d}{dx}h(x)\right). \end{aligned}$$

That is,

$$\frac{d}{dx}(f(x)\cdot g(x)\cdot h(x)) = f'(x)\cdot g(x)\cdot h(x) + f(x)\cdot g'(x)\cdot h(x) + f(x)\cdot g(x)\cdot h'(x).$$

Section 3.3: Derivatives of Trigonometric Functions

Problem 1. Differentiate.

(a) $y = \dfrac{t\,\sin(t)}{1+t}$ (b) $f(\theta) = \theta\cdot\cos(\theta)\cdot\sin(\theta)$

HINT: Use the Product Rule for three functions (shown above) in part (b).

(a)

$$\begin{aligned} y' &= \frac{(t\cos t + \sin t)(1+t) - (1)t\sin t}{(1+t)^2} \\ &= \frac{t\cos t + \sin t + t^2\cos t + t\sin t - t\sin t}{(1+t)^2} = \frac{(t^2+t)\cos t + \sin t}{(1+t)^2} \end{aligned}$$

(b)

$$\begin{aligned} f'(\theta) &= 1\cos\theta\,\sin\theta + \theta(-\sin\theta)\sin\theta + \theta\cos\theta(\cos\theta) \\ &= \cos\theta\,\sin\theta - \theta\sin^2\theta + \theta\cos^2\theta \end{aligned}$$

Problem 2. Find the x-values at which the tangent line is horizontal to the given curve when x satisfies $\pi \le x \le 3\pi/2$.

$$y = \frac{\cos(x)}{2+\sin(x)}$$

HINT: Use the trigonometric Pythagorean identity $\cos^2(x) + \sin^2(x) = 1$ to simplify the derivative and think about the Unit Circle.

We will find the derivative of y and set it equal to zero to determine the x-values at which the tangent line is horizontal. We have

$$y' = \frac{(-\sin(x))(2+\sin(x)) - \cos(x)\cdot\cos(x)}{(2+\sin(x))^2} = \frac{-2\sin(x) - (\sin^2(x)+\cos^2(x))}{(2+\sin(x))^2} = \frac{-2\sin(x) - 1}{(2+\sin(x))^2}$$

Solving for $y' = 0$, we have

$$\frac{-2\sin(x) - 1}{(2+\sin(x))^2} = 0 \quad \Rightarrow \quad -2\sin(x) - 1 = 0 \quad \Rightarrow \quad \sin(x) = -\frac{1}{2}$$

Since $\pi \leq x \leq 3\pi/2$, the equation $\sin(x) = -1/2$ is satisfied when $x = 7\pi/6$ radians (or equivalently, $210°$).

Section 3.4: The Chain Rule

Problem 3. Differentiate.

(a) $H(r) = \dfrac{(r^2-1)^3}{(2r+1)^5}$ (b) $F(t) = e^{t\sin(2t)}$ (c) $f(t) = \tan(\sec(\cos(t)))$

(a)

$$H'(r) = \frac{3(r^2-1)^2(2r)\cdot(2r+1)^5 - (r^2-1)^3\cdot 5(2r+1)^4(2)}{[(2r+1)^5]^2} = \frac{2(2r+1)^4(r^2-1)^2[3r(2r+1) - 5(r^2-1)]}{(2r+1)^{10}}$$

$$= \frac{2(r^2-1)^2(6r^2+3r-5r^2+5)}{(2r+1)^6} = \frac{2(r^2-1)^2(r^2+3r+5)}{(2r+1)^6}$$

(b)

$$F'(t) = e^{t\sin(2t)}\frac{d}{dt}(t\sin(2t)) = e^{t\sin(2t)}\left(\sin(2t) + 2t\cos(2t)\right)$$

(c)

$$f'(t) = \sec^2\left(\sec(\cos(t))\right)\frac{d}{dt}(\sec(\cos(t))) = \sec^2\left(\sec(\cos(t))\right)\sec(\cos(t))\tan(\cos(t))\frac{d}{dt}(\cos(t))$$

$$= \sec^2\left(\sec(\cos(t))\right)\sec(\cos(t))\tan(\cos(t))(-\sin(t))$$

Problem 4. Find the points at which the tangent line to the curve $y = \sqrt{1-x^2}$ is perpendicular to the line $x + y = 1$.

We have

$$\frac{dy}{dx} = \frac{1}{2}(1-x^2)^{-1/2}(-2x) = \frac{-x}{(1-x^2)^{1/2}}.$$

The line $x + y = 1$ can be rewritten as $y = -x + 1$ in slope-intercept form, allowing us to see that its slope is -1. Then we set the derivative equal to 1 to determine the x-values at which a tangent line to the curve is perpendicular to $x + y = 1$. We have

$$\frac{-x}{(1-x^2)^{1/2}} = 1 \Rightarrow -x = (1-x^2)^{1/2} \Rightarrow x^2 = 1 - x^2 \Rightarrow 2x^2 = 1 \Rightarrow x^2 = \frac{1}{2} \quad \Rightarrow \quad x = \pm\frac{1}{\sqrt{2}}$$

By plugging in $x = \pm\dfrac{1}{\sqrt{2}}$ into y we have

$$y = \sqrt{1 - \left(\pm\frac{1}{\sqrt{2}}\right)^2} = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

Then the points at which the tangent line to the curve $y = \sqrt{1-x^2}$ is perpendicular to the line $x + y = 1$ are $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$ and $\left(-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$.