Section 14.5: The Chain Rule

Problem 1. Find dz/dt in two ways:
(i) by substituting the expressions for x and y and then using the Chain Rule,
(ii) by using the Chain Rule.

$$z = xye^y, \quad x = t^2, \quad y = 5t.$$

Do your answers (i) and (ii) agree?

(i) Substituting $x = t^2$ and $y = 5t$, we have

$$z = t^2(5t)e^{5t} = 5t^3e^{5t}.$$

Then

$$\frac{dz}{dt} = \frac{d}{dt}(5t^3e^{5t}) = 5\frac{d}{dt}(t^3e^{5t}) = 5\left(\frac{d}{dt}(t^3)\, e^{5t} + t^3\frac{d}{dt}e^{5t}\right) = 5\left(3t^2e^{5t} + t^3(5e^{5t})\right) = 15t^2e^{5t} + 25t^3e^{5t}.$$

(ii) Let us apply the Chain Rule first. We have

$$\begin{aligned}\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} + \frac{dz}{dy}\frac{dy}{dt} &= (ye^y)(2t) + (xe^y + xye^y)\,(5)\\ &= (5te^{5t})(2t) + 5(t^2e^{5t} + t^2(5t)e^{5t})\\ &= 10t^2e^{5t} + 5t^2e^{5t} + 25t^3e^{5t}\\ &= 15t^2e^{5t} + 25t^3e^{5t}.\end{aligned}$$

Yes, the answers agree!

Problem 2. Let $z = \tan\left(\frac{u}{v}\right), \quad u = 7s + 3t, \quad v = 3s - 7t$. Find $\partial z/\partial s$ and $\partial z/\partial t$.

Let's rewrite $z = \tan\left(\frac{u}{v}\right) = \tan\left(uv^{-1}\right)$. Then

$$\begin{aligned}\frac{\partial z}{\partial s} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial s} &= \sec^2(uv^{-1})\,(v^{-1})\,\frac{\partial}{\partial s}(7s + 3t) + \sec^2(uv^{-1})\,(-uv^{-2})\,\frac{\partial}{\partial s}(3s - 7t)\\ &= 7\sec^2\left(\frac{u}{v}\right)\frac{1}{v} + 3\sec^2\left(\frac{u}{v}\right)\left(\frac{-u}{v^2}\right),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial z}{\partial t} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t} &= \sec^2(uv^{-1})\,(v^{-1})\,\frac{\partial}{\partial t}(7s + 3t) + \sec^2(uv^{-1})\,(-uv^{-2})\,\frac{\partial}{\partial t}(3s - 7t)\\ &= 3\sec^2\left(\frac{u}{v}\right)\frac{1}{v} - 7\sec^2\left(\frac{u}{v}\right)\left(\frac{-u}{v^2}\right).\end{aligned}$$

Problem 3. Let $R(s,t) = G(u(s,t), v(s,t))$, where G, u, and v are differentiable and the following applies:

$$\begin{aligned} u(-2,-6) &= 2 \qquad & v(-2,-6) &= 7, \\ u_s(-2,-6) &= 5 \qquad & v_s(-2,-6) &= -9, \\ u_t(-2,-6) &= -7 \qquad & v_t(-2,-6) &= 9, \\ G_u(2,7) &= 3 \qquad & G_v(2,7) &= -3. \end{aligned}$$

Find $R_s(-2,-6)$ and $R_t(-2,-6)$.

Let's make this problem look simpler. Let $z = R(s,t)$. Then $z = G(u,v)$, where $u = u(s,t)$ and $v = v(s,t)$. We have

$$\begin{aligned} R_s(s,t) = \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial s} \\ &= \frac{\partial G}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial G}{\partial v}\frac{\partial v}{\partial s}, \quad \text{since } z = G(u,v), \\ &= G_u(u,v)u_s(s,t) + G_v(u,v)v_s(s,t). \end{aligned}$$

Note that we are given that when $s = -2$ and $t = -6$, $u = u(-2,-6) = 2$ and $v = v(-2,-6) = 7$. Then by the above,

$$R_s(-2,-6) = G_u(2,7)u_s(-2,-6) + G_v(2,7)v_s(-2,-6) = 3(5) + (-3)(-9) = 15 + 27 = 42.$$

Similarly,

$$R_t(s,t) = G_u(u,v)u_t(s,t) + G_v(u,v)v_t(s,t)$$

$$\Downarrow$$

$$R_t(-2,-6) = G_u(2,7)u_t(-2,-6) + G_v(2,7)v_t(-2,-6) = (3)(-7) + (-3)(9) = -48.$$

Problem 4. The radius of a right circular cone is increasing at a rate of 1.4 cm/s while its height is decreasing at a rate of 2.8 cm/s. At what rate is the volume of the cone changing when the radius is 115 cm and the height is 124 cm?

Hint: The volume of a right circular cone is $V = \dfrac{\pi r^2 h}{3}$.

Let r be the radius of the right circular cone and let h be its height. Note that $r = r(t)$ and $h = h(t)$ are both functions of time (t is for time) and their units are in cm. The units of time are seconds.

GIVEN: $\dfrac{dr}{dt}$ = rate of change of r with respect to time $= 1.4\, cm/s$,

$\dfrac{dh}{dt}$ = rate of change of h with respect to time $= 2.8\, cm/s$,

The volume of a right circular cone is $V = \dfrac{\pi r^2 h}{3}$.

GOAL: Find $\dfrac{dV}{dt}$ = rate of change of V with respect to time when $r = 115$ and $h = 124$.

Since $V = V(r, h)$ is a function of r and h, by the Chain Rule, we have

$$\begin{aligned}
\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} + \frac{dV}{dh}\frac{dh}{dt} &= \left(\frac{2\pi}{3} rh\right)\frac{dr}{dt} + \left(\frac{\pi r^2}{3}\right)\frac{dh}{dt} \\
&= \left(\frac{2\pi}{3}(115)(124)\right)(1.4) + \left(\frac{\pi(115)^2}{3}\right)(2.8) \\
&= \left(\left(\frac{2}{3}(115)(124)\right)(1.4) + \left(\frac{(115)^2}{3}\right)(2.8)\right)\pi \\
&= 966\pi\ cm^3/s.
\end{aligned}$$

Problem 5. The temperature at a point (x, y) is $T(x, y)$, measured in degrees Celsius. A bug crawls so that its position after t seconds is given by $x = \sqrt{1+t}$, $y = 2 + \frac{1}{3}t$, where x and y are measured in centimeters. The temperature function satisfies

$$T_x(2,3) = 9 \qquad \text{and} \qquad T_y(2,3) = 2.$$

How fast is the temperature rising on the bug's path after 3 seconds? (Round your answer to two decimal places.)

GIVEN: $x = (1+t)^{1/2}$ and $y = 2 + \dfrac{1}{3}t$, both of which are measured in cm,
$T_x(2,3) = 9\,^\circ C/cm$,
$T_y(2,3) = 2\,^\circ C/cm$.

GOAL: Find $\dfrac{dT}{dt}$ = rate of change of T with respect to time when $t = 3$.

Since $T = T(x, y)$ is a function of x and y, by the Chain Rule, we have

$$\frac{dT}{dt} = \frac{dT}{dx}\frac{dx}{dt} + \frac{dT}{dy}\frac{dy}{dt}. \tag{1}$$

Note that

$$\frac{dx}{dt} = \frac{1}{2}(1+t)^{-1/2} \quad \Rightarrow \frac{dx}{dt}\bigg|_{t=3} = \frac{1}{2\sqrt{1+3}} = \frac{1}{4}$$

and

$$\frac{dy}{dt} = \frac{1}{3} \quad \Rightarrow \frac{dy}{dt}\bigg|_{t=3} = \frac{1}{3}.$$

Also,

$$x(3) = \sqrt{1+3} = 2 \quad \text{and} \quad y(3) = 2 + \frac{3}{3} = 3.$$

Combining this with (1), we have

$$\frac{dT}{dt}\bigg|_{t=3} = T_x(2,3)\,\frac{dx}{dt}\bigg|_{t=3} + T_y(2,3)\,\frac{dy}{dt}\bigg|_{t=3} = 9\left(\frac{1}{4}\right) + 2\left(\frac{1}{3}\right) \approx 2.92\,^\circ C/s.$$