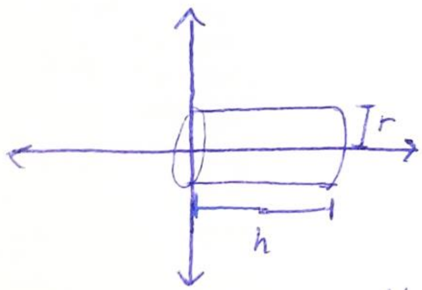
MAT 1505 (Dr. Fuentes) Worksheet 4 - Section 6.2

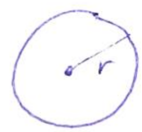
Section 6.2: Volumes

Problem 1.
(a) Show, using an integral, that the volume of a circular cylinder with height h and base radius r is $V = \pi r^2 h$.
(b) Find, using an integral, the volume of a right circular cone with height h and base radius r.

(a)



Cross-section

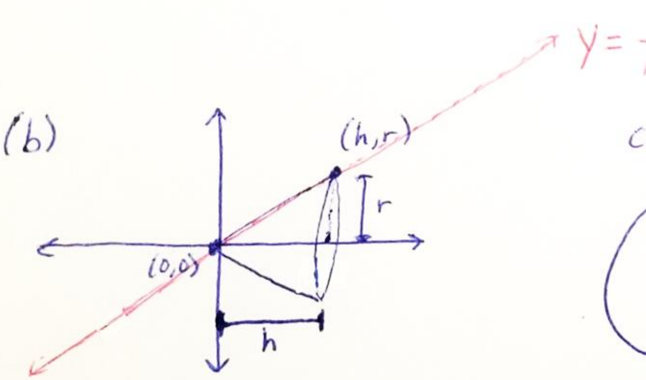


Area $A(x) = \pi r^2$

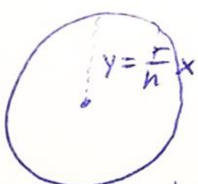
$$\text{Volume} = \int_0^h A(x)\,dx = \int_0^h \pi r^2\,dx = \pi r^2 \int_0^h 1\,dx = \pi r^2 x \Big]_0^h$$

$$= \pi r^2 h$$

(b)



Cross-section



Area $A(x) = \pi\left(\frac{r}{h}x\right)^2 = \frac{\pi r^2}{h^2}x^2$

$$\text{Volume} = \int_0^h \frac{\pi r^2}{h^2}x^2\,dx = \frac{\pi r^2}{h^2}\int_0^h x^2\,dx$$

$$= \frac{\pi r^2}{h^2}\cdot\frac{1}{3}x^3\Big]_0^h$$

$$= \frac{\pi r^2}{h^2}\cdot\frac{1}{3}\cdot h^3$$

$$= \frac{1}{3}\pi r^2 h$$

Problem 2. Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.
Sketch the region, the 3-dimensional solid formed by rotating the region 360° about the line, and a typical disk or washer.

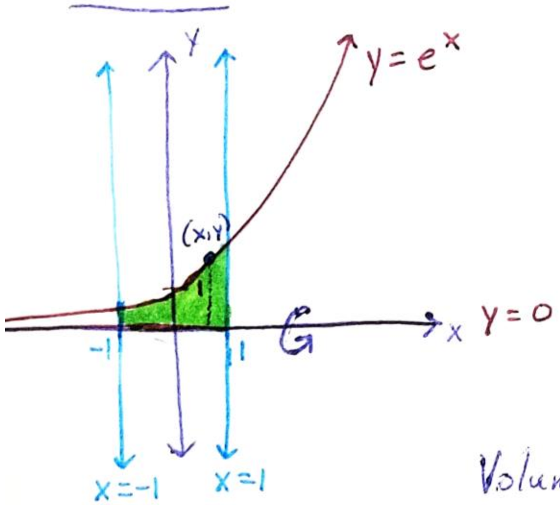
(a) $y = e^x$, $y = 0, x = -1$, and $x = 1$; about the x-axis,

(b) $2x = y^2$, $x = 0, y = 4$; about the y-axis,

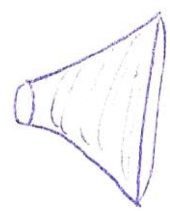
(c) $y = x^3$, $y = 1, x = 2$; about $y = -3$.

(a)

REGION



SOLID



CROSS-SECTION

AREA $A(x) = \pi (e^x)^2$
$= \pi e^{2x}$

$$\text{Volume} = \pi \int_{-1}^{1} e^{2x}\,dx$$

Let
$u = 2x \Rightarrow du = 2\,dx$
$\frac{1}{2} du = dx$

When
$x = 1 \Rightarrow u = 2 \cdot 1 = 2$
$x = -1 \Rightarrow u = 2(-1) = -2$

$$= \pi \int_{-2}^{2} e^u \left(\tfrac{1}{2} du\right)$$

$$= \frac{\pi}{2} \int_{-2}^{2} e^u du = \frac{\pi}{2} e^u \Big]_{-2}^{2}$$

$$= \frac{\pi}{2}\left(e^2 - e^{-2}\right)$$

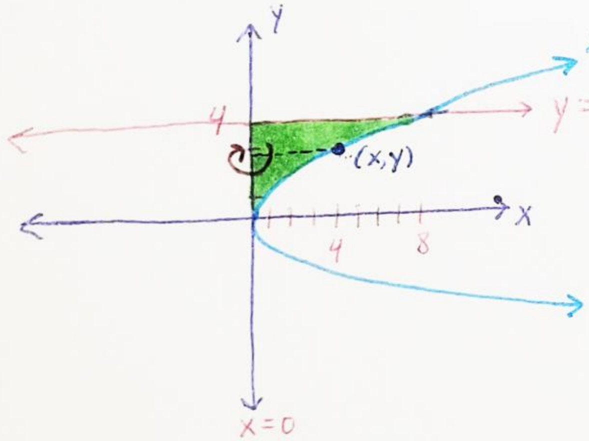
Since integral is of the form $\int_{-a}^{a} f(x)\,dx$, check if A is even/odd/neither.

$A(-x) = \pi e^{2(-x)}$
$= \pi e^{-2x}$
not equal to
$A(x)$ nor $-A(x) = -\pi e^{2x}$,
so A is neither odd nor even.

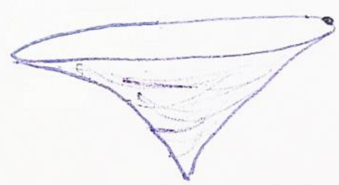
(b) Since the region is being rotated about the y-axis, we will need to integrate with respect to y, which means that $2x = y^2$ must be rewritten as

$x = \frac{1}{2}y^2$, so that x is a function of y.

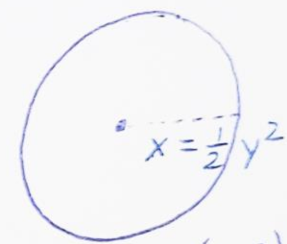
REGION



SOLID



CROSS-SECTION



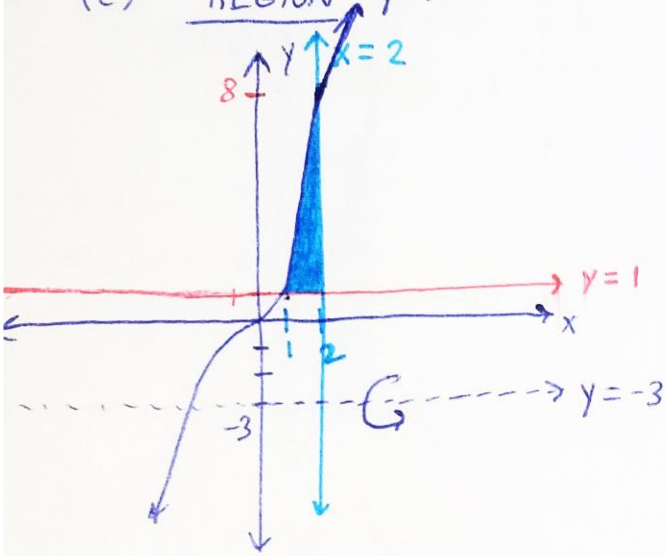
AREA $A(y) = \pi\left(\frac{1}{2}y^2\right)^2$

$= \frac{\pi}{4}y^4$

$$\text{Volume} = \int_0^4 A(y)\,dy$$

$$= \frac{\pi}{4}\int_0^4 y^4\,dy = \frac{\pi}{4}\cdot\frac{1}{5}y^5\Big]_0^4 = \frac{\pi}{20}4^5 = \frac{256\pi}{20}$$

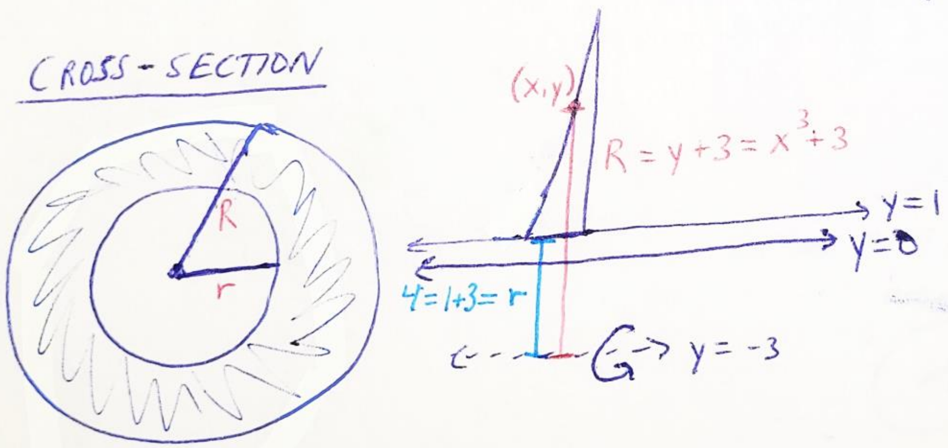
(c) REGION



SOLID

* Integrate with respect to x since solid is being rotated about a horizontal axis.

CROSS-SECTION



AREA $A(x) = \pi R^2 - \pi r^2$

$= \pi(R^2 - r^2)$

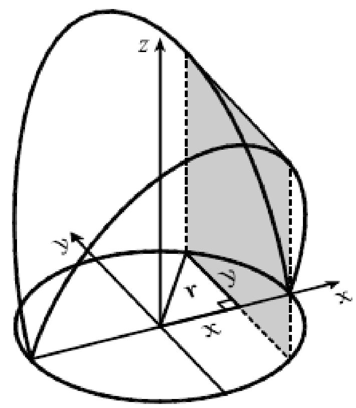
$= \pi\left((x^3+3)^2 - 4^2\right) = \pi\left(x^6 + 6x^3 + 9 - 16\right) = \pi\left(x^6 + 6x^3 - 7\right).$

$$\text{Volume} = \int_1^2 A(x)\,dx = \pi \int_1^2 (x^6 + 6x^3 - 7)\,dx$$

$$= \pi\left(\frac{1}{7}x^7 + \frac{3}{2}x^4 - 7x\right)\Big]_1^2$$

$$= \pi\left(\frac{1}{7}2^7 + \frac{3}{2}2^4 - 7\cdot 2 - \left(\frac{1}{7} + \frac{3}{2} - 7\right)\right) = \frac{471\pi}{14}$$

Problem 3. Find the volume of the following solid S: The base of S is a circular disk with radius r. Parallel cross-sections perpedicular to the base are squares.



A cross-section is shaded in the diagram.

$A(x) = (2y)^2 = \left(2\sqrt{r^2 - x^2}\right)^2$, so

$$V = \int_{-r}^{r} A(x)\,dx = 2\int_0^r 4(r^2 - x^2)\,dx$$

$$= 8\left[r^2 x - \tfrac{1}{3}x^3\right]_0^r = 8\left(\tfrac{2}{3}r^3\right) = \tfrac{16}{3}r^3$$