MAT 1500 (Dr. Fuentes) **Worksheet 5 - Sections 2.8, 3.1, & 3.2**

Section 2.8: The Derivative as a Function

Problem 1.
(a) Show that $f(x) = x^{2/3}$ is not differentiable at $x = 0$.

(b) Determine where the function $g(x) = x + |x|$ is not differentiable. Draw the graphs of g and g'.

(a) We have

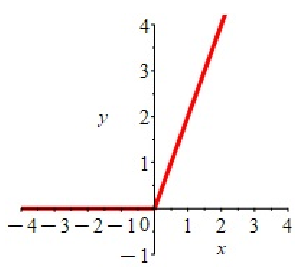
$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{(0+h)^{2/3} - 0^{2/3}}{h} = \lim_{h\to 0} \frac{h^{2/3}}{h} = \lim_{h\to 0} \frac{1}{h^{1/3}}.$$

As $h \to 0^+$, we see that $\frac{1}{h^{1/3}} \to \infty$ and as $h \to 0^-$, we see that $\frac{1}{h^{1/3}} \to -\infty$. Therefore, the limit above does not exist. That is, $f'(0)$ does not exist, which means that f is not differentiable at $x = 0$.

(b) We can express the function g as a piecewise function as follows:

$$g(x) = \begin{cases} x + x & \text{if } x \geq 0 \\ x - x & \text{if } x < 0 \end{cases} = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

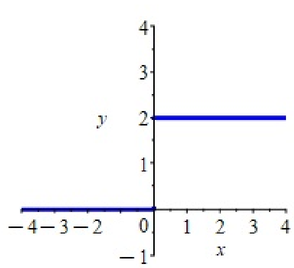
The graph of g looks like



The curve of g appears to be differentiable everywhere except at $x = 0$, where we see a corner. Therefore, $g'(0)$ does not exist; that is, g is not differentiable at $x = 0$. The derivative of g is

$$g'(x) = \begin{cases} 2 & \text{if } x > 0 \\ 0 & \text{if } x < 0, \end{cases}$$

and the graph of g' looks like



Problem 2. Let

$$f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 5 - x & \text{if } 0 < x < 4 \\ \dfrac{1}{5-x} & \text{if } x \ge 4 \end{cases}$$

(a) Where is f discontinuous?
(b) Find $f'(4)$, if it exists. If it does not exist, show why.
(c) Where is f differentiable?

(a) We see that f is discontinuous at $x = 5$ since $f(5)$ is undefined. Notice that

$$\lim_{x\to 0^-} f(x) = 0 \quad \text{and} \quad \lim_{x\to 0^+} f(x) = 5 - 0 = 0 \quad \Rightarrow \lim_{x\to 0} f(x)\ DNE.$$

Then f is also discontinuous at 0. However,

$$\lim_{x\to 4^-} f(x) = 5 - 4 = 1 = f(4) = \lim_{x\to 4^+} f(x) = \frac{1}{5-4},$$

meaning that f is continuous at $x = 4$.

In summary, f is discontinuous at 0 and at 5.

(b) We have

$$\lim_{x\to 4-} \frac{f(x) - f(4)}{x - 4} = \lim_{x\to 4^-} \frac{(5-x) - 1}{x - 4} = \lim_{x\to 4^-} \frac{4 - x}{x - 4} == \lim_{x\to 4^-} -1 = -1$$

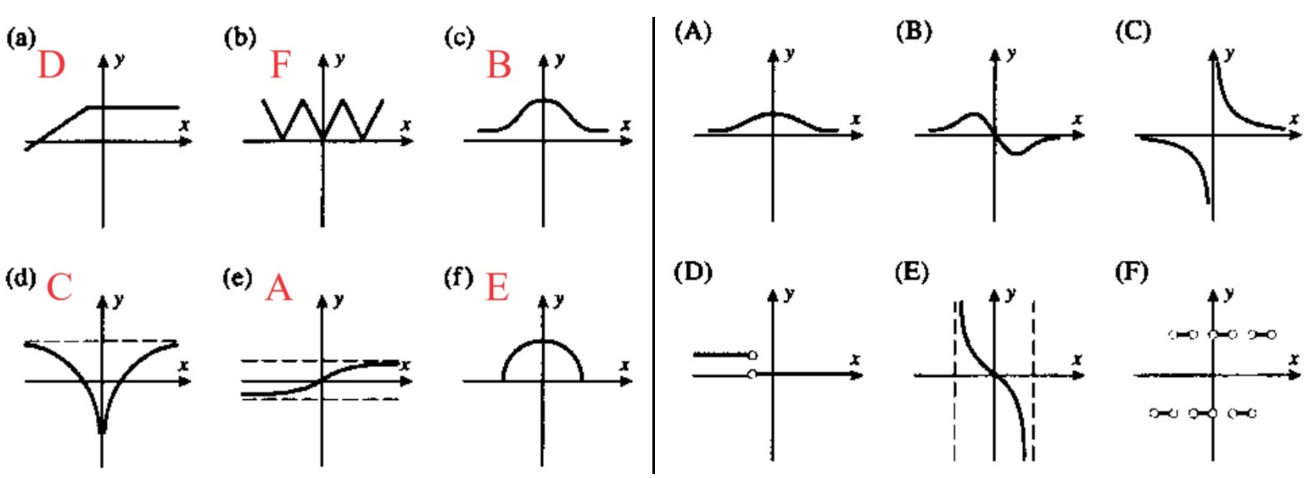
and

$$\lim_{x\to 4^+} \frac{f(x) - f(4)}{x - 4} = \lim_{x\to 4^+} \frac{\frac{1}{5-x} - 1}{x - 4} = \lim_{x\to 4^+} \frac{\frac{1-(5-x)}{5-x}}{x - 4} = \lim_{x\to 4^+} \frac{\frac{x-4}{5-x}}{x - 4} = \lim_{x\to 4^+} \frac{1}{5-x} = \frac{1}{5-4} = 1.$$

Since the previous two limits are unequal, then $f'(4) = \lim_{x\to 4} \dfrac{f(x) - f(4)}{x - 4}$ does not exist, meaning that f is not differentiable at 4.

(c) Since f is discontinuous at 0 and at 5, then f is not differentiable at 0 and at 5. In part (b), we showed that f is not differentiable at 4. Therefore, f is not differentiable at 0, 4, and 5.

Problem 3. Match the graphs of the functions in (a)-(f) with the graphs of their derivatives in (A)-(F).



Section 3.1: Derivatives of Polynomials & Exponential Functions

Problem 4. Problem 4. Find the point on the curve $y = 1 + 2e^x - 3x$ at which the tangent line is parallel to the line $3x - y = 5$.

We have

$$\frac{dy}{dx} = 0 + 2e^x - 3 = 2e^x - 3.$$

The slope of the line $3x - y = 5$ is 3. Then a line that is parallel to this line will also have slope 3. Then we solve $\frac{dy}{dx} = 3$ to find the x-value(s) at which the tangent line to the given curve is perpendicular to the given line. We have

$$2e^x - 3 = 3 \quad \Rightarrow \quad 2e^x = 6 \quad \Rightarrow \quad e^x = 3 \quad \Rightarrow \quad x = \ln(3).$$

By plugging in $x = \ln(3)$ into $y = 1 + 2e^x - 3x$, we have

$$y = 1 + 2e^{\ln(3)} - 3 \cdot \ln(3) = 1 + 2 \cdot 3 - 3\ln(3) = 7 - 3\ln(3).$$

Therefore, the point at which the tangent line is parallel to the line $3x - y = 5$ is $(\ln(3), 7 - 3\ln(3))$.

Problem 5. Show that the curve $y = 2e^x + 3x + 5x^3$ has no tangent line with slope 2.

We have

$$\frac{dy}{dx} = 2e^x + 3 + 15x^2.$$

If

$$2e^x + 3 + 15x^2 = 2 \quad \Rightarrow \quad 2e^x + 15x^2 = -1.$$

However, since $2e^x > 0$ and $15x^2 \geq 0$, this means we have

$$0 < 2e^x + 15x^2 = -1 \quad \Rightarrow 0 < -1,$$

which is impossible! This means that no tangent line to the curve of y can have slope 2.

Section 3.2: The Product & Quotient Rules

Problem 6. Find $f'(x)$ and $f''(x)$ for $f(x) = \sqrt{x}e^x$.

$$f(x) = \sqrt{x}\,e^x \quad \overset{\text{PR}}{\Rightarrow} \quad f'(x) = \sqrt{x}\,e^x + e^x\left(\frac{1}{2\sqrt{x}}\right) = \left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right)e^x = \frac{2x+1}{2\sqrt{x}}\,e^x.$$

Using the Product Rule and $f'(x) = \left(x^{1/2} + \frac{1}{2}x^{-1/2}\right)e^x$, we get

$$f''(x) = \left(x^{1/2} + \tfrac{1}{2}x^{-1/2}\right)e^x + e^x\left(\tfrac{1}{2}x^{-1/2} - \tfrac{1}{4}x^{-3/2}\right) = \left(x^{1/2} + x^{-1/2} - \tfrac{1}{4}x^{-3/2}\right)e^x = \frac{4x^2 + 4x - 1}{4x^{3/2}}\,e^x$$

Problem 7. Find the derivative of $y = \dfrac{x^2 e^x}{x^2 + e^x}$.

$$f(x) = \frac{x^2 e^x}{x^2 + e^x} \quad \overset{\text{QR}}{\Rightarrow}$$

$$f'(x) = \frac{(x^2 + e^x)\left[x^2 e^x + e^x(2x)\right] - x^2 e^x(2x + e^x)}{(x^2 + e^x)^2} = \frac{x^4 e^x + 2x^3 e^x + x^2 e^{2x} + 2xe^{2x} - 2x^3 e^x - x^2 e^{2x}}{(x^2 + e^x)^2}$$

$$= \frac{x^4 e^x + 2xe^{2x}}{(x^2 + e^x)^2} = \frac{xe^x(x^3 + 2e^x)}{(x^2 + e^x)^2}$$