Section 7.8: Trigonometric Integrals

TYPE 1 Improper Integrals are of the form

$$\int_a^{\infty} f(x)\,\mathrm{dx}, \quad \int_{-\infty}^{b} f(x)\,\mathrm{dx}, \quad \text{or} \quad \int_{-\infty}^{\infty} f(x)\,\mathrm{dx}$$

Problem 1. Determine whether the integral is convergent or divergent. Evaluate the integrals thst are convergent.

(a) $\displaystyle\int_{-\infty}^{0} \frac{x}{(x^2+1)^3}\,\mathrm{dx}$, (b) $\displaystyle\int_{0}^{\infty} \sin(\theta)e^{\cos(\theta)}\,\mathrm{d}\theta$, (c) $\displaystyle\int_{1}^{\infty} \frac{\ln(x)}{x^2}\,\mathrm{dx}$, (d) $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\,\mathrm{dx}$

(a) We have

$$\int_{-\infty}^{0} \frac{x}{(x^2+1)^3}\,\mathrm{dx} = \lim_{t\to-\infty}\int_{t}^{0} \frac{x}{(x^2+1)^3}\,\mathrm{dx}$$

Let us compute the definite integral $\int_t^0 \frac{x}{(x^2+1)^3}\,\mathrm{dx}$. Let

$$u = x^2+1. \qquad \text{Then} \qquad du = 2\,dx \Rightarrow \frac{1}{2}du = dx,$$

and when

$$x = 0 \quad \Rightarrow \quad u = 0^2+1 = 1$$

and when

$$x = t \quad \Rightarrow \quad u = t^2+1.$$

Then

$$\int_{t}^{0} \frac{x}{(x^2+1)^3}\,\mathrm{dx} = \int_{t^2+1}^{1} \frac{1}{u^3}\left(\frac{1}{2}\mathrm{du}\right) = \frac{1}{2}\int_{t^2+1}^{1} u^{-3}\,\mathrm{du} = -\frac{1}{4}u^{-2}\Big]_{t^2+1}^{1} = -\frac{1}{4}\left(1-\frac{1}{(t^2+1)^2}\right).$$

Then we have

$$\int_{-\infty}^{0} \frac{x}{(x^2+1)^3}\,\mathrm{dx} = \lim_{t\to-\infty} -\frac{1}{4}\left(1-\frac{1}{(t^2+1)^2}\right) = -\frac{1}{4}\lim_{t\to-\infty}\left(1-\frac{1}{(t^2+1)^2}\right) = -\frac{1}{4}(1-0) = -\frac{1}{4}.$$

Therefore, the integral is **convergent**.

(b) We have

$$\int_{0}^{\infty} \sin(\theta)\,e^{\cos(\theta)}\,\mathrm{d}\theta = \lim_{t\to\infty}\int_{0}^{t} \sin(\theta)\,e^{\cos(\theta)}\,\mathrm{d}\theta$$

Let us compute the definite integral $\int_0^t \sin(\theta)\,e^{\cos(\theta)}\,\mathrm{d}\theta$. Let

$$u = \cos(\theta). \qquad \text{Then} \qquad -du = \sin(\theta)\,\mathrm{d}\theta,$$

and when

$$\theta = t \quad \Rightarrow \quad u = \cos(t)$$

and when

$$\theta = 0 \quad \Rightarrow \quad u = \cos(0) = 1.$$

Then

$$\int_0^t \sin(\theta)\, e^{\cos(\theta)}\, \mathrm{d}\theta = -\int_1^{\cos(t)} e^u\, \mathrm{du} = -e^u\Big]_1^{\cos(t)} = -\left(e^1 - e^{\cos(t)}\right) = e^{\cos(t)} - e.$$

Then we have

$$\int_0^\infty \sin(\theta)\, e^{\cos(\theta)}\, \mathrm{d}\theta = \lim_{t\to\infty} \left(e^{\cos(t)} - e\right).$$

However, since $\cos(t)$ oscillates infinitely many times between -1 and 1 as $t \to \infty$, then $e^{\cos(t)}$ oscillates infinitely many times between e^{-1} and e as $t \to \infty$. Therefore the **limit** $\lim_{\mathbf{t}\to\infty} \left(\mathbf{e}^{\cos(\mathbf{t})} - \mathbf{e}\right)$ **DNE, meaning the integral is divergent.**

(c) We have

$$\int_1^\infty \frac{\ln(x)}{x^2}\, \mathrm{dx} = \lim_{t\to\infty} \int_1^t \frac{\ln(x)}{x^2}\, \mathrm{dx}.$$

Let us compute the definite integral $\int_1^t \frac{\ln(x)}{x^2}\, \mathrm{dx}$.

Note: The function $f(x) = \ln(x)/x^2$ is not a familiar antiderivative of a function, nor can we apply a u-substitution, since the derivatives of $\ln(x)$ and x^2 do not help us substitute the rest of the integral. Since the function $f(x) = \ln(x)/x^2$ is the product of $\ln(x)$ and $1/x^2$, **integration by parts seems appropriate.

Let

$$u = \ln(x) \quad \text{and} \quad \mathrm{dv} = \frac{1}{x^2}\, \mathrm{dx} \quad \Rightarrow \quad \mathrm{du} = \frac{1}{x}\, \mathrm{dx} \quad \text{and} \quad v = \int x^{-2}\, \mathrm{dx} = -\frac{1}{x}\, \mathrm{dx}.$$

Then

$$\begin{aligned}\int_1^t \frac{\ln(x)}{x^2}\, \mathrm{dx} = \int_1^t u\, \mathrm{dv} = uv\Big]_1^t - \int_1^t v\, \mathrm{du} = \frac{\ln(x)}{x}\Big]_1^t - \int_1^t -\frac{1}{x^2}\, \mathrm{du} &= \left(\ln(1) - \frac{\ln(t)}{t}\right) - \left(\frac{1}{x}\right)\Big]_1^t \\ &= -\frac{\ln(t)}{t} - \left(1 - \frac{1}{t}\right) \\ &= -\frac{\ln(t)}{t} - 1 + \frac{1}{t}.\end{aligned}$$

Then we have

$$\begin{aligned}\int_1^\infty \frac{\ln(x)}{x^2}\, \mathrm{dx} &= \lim_{t\to\infty} -\frac{\ln(t)}{t} - 1 + \frac{1}{t} \\ &= \lim_{t\to\infty} -\frac{\ln(t)}{t} - \lim_{t\to\infty} 1 + \lim_{t\to\infty} \frac{1}{t} \\ &= \lim_{t\to\infty} -\frac{\frac{1}{t}}{1} - 1 + 0, \quad \text{by L'Hospital's Rule since } \frac{-\ln(t)}{t} \text{ is in } \frac{\infty}{\infty} \text{ indeterminate form} \\ &= 0 - 1 + 0 = -1.\end{aligned}$$

Therefore, the integral is convergent.

(d) We rewrite

$$\int_{-\infty}^{\infty} xe^{-x^2}\,\mathrm{dx} = \int_{-\infty}^{0} xe^{-x^2}\,\mathrm{dx} + \int_{0}^{\infty} xe^{-x^2}\,\mathrm{dx}.$$

Let us compute $I_1 = \int_{-\infty}^{0} xe^{-x^2}\,\mathrm{dx}$ first. We have

$$\int_{-\infty}^{0} xe^{-x^2}\,\mathrm{dx} = \lim_{t\to-\infty} \int_{t}^{0} xe^{-x^2}\,\mathrm{dx}.$$

We focus our attention to finding $\int_{t}^{0} xe^{-x^2}\,\mathrm{dx}$. We can try a u-substitution. Let

$$u = -x^2 \qquad \Rightarrow \qquad \mathrm{du} = -2x\,\mathrm{dx} \qquad \Rightarrow \qquad -\frac{1}{2}\mathrm{du} = x\,\mathrm{dx}.$$

Substituting the limits of integration, we have that when

$$x = 0 \quad \Rightarrow \quad u = -0^2 = 0 \quad \text{and when} \quad x = t \quad \Rightarrow \quad u = -t^2.$$

Then

$$\int_{t}^{0} xe^{-x^2}\,\mathrm{dx} = -\frac{1}{2}\int_{-t^2}^{0} e^{u}\,\mathrm{du} = -\frac{1}{2}\left(e^{u}\right)\Big]_{-t^2}^{0} = -\frac{1}{2}\left(1 - \frac{1}{e^{t^2}}\right).$$

Then the integral I_1 becomes

$$I_1 = \int_{-\infty}^{0} xe^{-x^2}\,\mathrm{dx} = \lim_{t\to-\infty} -\frac{1}{2}\left(1 - \frac{1}{e^{t^2}}\right) = -\frac{1}{2}\left(1 - 0\right) = -\frac{1}{2}.$$

Therefore, I_1 is covergent. This means we will have to find out whether $I_2 = \int_{0}^{\infty} xe^{-x^2}\,\mathrm{dx}$ is convergent or divergent.

Note that

$$\begin{aligned} I_2 = \int_{0}^{\infty} xe^{-x^2}\,\mathrm{dx} = \lim_{t\to\infty} \int_{0}^{t} xe^{-x^2}\,\mathrm{dx} &= \lim_{t\to\infty} - \int_{t}^{0} xe^{-x^2}\,\mathrm{dx} \\ &- \lim_{t\to\infty} \int_{t}^{0} xe^{-x^2}\,\mathrm{dx} \\ &- \lim_{t\to\infty} \left(-\frac{1}{2}\left(1 - \frac{1}{e^{t^2}}\right)\right) \quad \text{by our previous work for computing } I_1 \\ &= -\left(-\frac{1}{2}\right) = \frac{1}{2}. \end{aligned}$$

Then we have

$$\int_{-\infty}^{\infty} xe^{-x^2}\,\mathrm{dx} = I_1 + I_2 = -\frac{1}{2} + \frac{1}{2} = 0,$$

meaning the integral is convergent.