Section 7.2: Trigonometric Integrals

Trigonometric integrals may require the use of the following trigonometric identities (these will be provided on quizzes and exams):

$$\cos^2(\mathbf{x}) + \sin^2(\mathbf{x}) = \mathbf{1}, \tag{1}$$

$$\cos^2(\mathbf{x}) = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{1} + \cos(\mathbf{2x})), \tag{2}$$

$$\sin^2(\mathbf{x}) = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{1} - \cos(\mathbf{2x})). \tag{3}$$

Problem 1. Evaluate the following integrals:
(a) $\displaystyle\int \frac{\sin^2(1/t)}{t^2}\,\mathrm{dt}$, (b) $\displaystyle\int \cos^3(t/2)\,\sin^2(t/2)\,\mathrm{dt}$, (c) $\displaystyle\int_0^{\pi/4} \frac{\sin^3(x)}{\cos(x)}\,\mathrm{dx}$.

(a) We begin by applying a u-substitution. Let

$$u = 1/t = t^{-1}.$$

Then

$$du = -t^{-2}\,dt = -1/t^2\,dt,$$

or equivalently,

$$-du = 1/t^2\,dt.$$

Then

$$\begin{aligned}
\int \frac{\sin^2(1/t)}{t^2}\,\mathrm{dt} = \int \sin^2(u)\,(-\mathrm{du}) &= -\int \frac{1}{2}(1-\cos(2u))\,\mathrm{du}, \quad \text{by (3)},\\
&= -\frac{1}{2}\int (1-\cos(2u))\,\mathrm{du}\\
&= -\frac{1}{2}\left(u - \frac{1}{2}\sin(2u)\right) + C\\
&= -\frac{1}{2}\left(\frac{1}{t} - \frac{1}{2}\sin\left(\frac{2}{t}\right)\right) + C\\
&= -\frac{1}{2t} + \frac{1}{4}\sin\left(\frac{2}{t}\right) + C.
\end{aligned}$$

(b) Since the integral has an odd power of $\cos(x)$, we rewrite the integral as follows:

$$\int \cos^3\left(\frac{t}{2}\right)\sin^2\left(\frac{t}{2}\right)\,\mathrm{dt} = \int \cos^2\left(\frac{t}{2}\right)\sin^2\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)\,\mathrm{dt} = \int \left(1-\sin^2\left(\frac{t}{2}\right)\right)\sin^2\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)\,\mathrm{dt}.$$

We apply a u-substitution. Let

$$u = \sin(t/2).$$

Then

$$du = \frac{1}{2}\cos\left(\frac{t}{2}\right)\,dt,$$

or equivalently,

$$2\,du = \cos\left(\frac{t}{2}\right)\,dt.$$

Then

$$\begin{aligned}
\int \cos^3\left(\frac{t}{2}\right)\sin^2\left(\frac{t}{2}\right)\,\mathrm{dt} = \int \left(1-\sin^2\left(\frac{t}{2}\right)\right)\sin^2\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)\,\mathrm{dt} &= \int (1-u^2)u^2\,(2\,du) \\
&= 2\int (u^2-u^4)\,du \\
&= 2\left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right) + C \\
&= 2\left(\frac{1}{3}\sin^3\left(\frac{t}{2}\right) - \frac{1}{5}\sin^5\left(\frac{t}{2}\right)\right) + C \\
&= \frac{2}{3}\sin^3\left(\frac{t}{2}\right) - \frac{2}{5}\sin^5\left(\frac{t}{2}\right) + C.
\end{aligned}$$

(c) Since the integral has an odd power of $\sin(x)$, we rewrite the integral as follows:

$$\int_0^{\pi/4} \frac{\sin^3(x)}{\cos(x)}\,\mathrm{dx} = \int_0^{\pi/4} \frac{\sin^2(x)}{\cos(x)}\sin(x)\,\mathrm{dx} = \int_0^{\pi/4} \frac{1-\cos^2(x)}{\cos(x)}\sin(x)\,\mathrm{dx}.$$

We apply a u-substitution. Let

$$u = \cos(x).$$

Then

$$-du = \sin(x)\,\mathrm{dx}.$$

Since we are computing a definite integral, we must also substitute the limits of integration. When

$$x = 0, \implies u = \cos(0) = 1, \quad \text{and when} \quad x = \frac{\pi}{4}, \implies u = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

Then

$$\begin{aligned}
\int_0^{\pi/4} \frac{\sin^3(x)}{\cos(x)}\,\mathrm{dx} = \int_0^{\pi/4} \frac{1-\cos^2(x)}{\cos(x)}\sin(x)\,\mathrm{dx} &= \int_1^{\frac{\sqrt{2}}{2}} \frac{1-u^2}{u}\,(-\mathrm{du}) \\
&= \int_{\frac{\sqrt{2}}{2}}^1 \frac{1-u^2}{u}\,\mathrm{du} \\
&= \int_{\frac{\sqrt{2}}{2}}^1 (u^{-1}-u)\,\mathrm{du} \\
&= \left(\ln(|u|) - \frac{1}{2}u^2\right)\Big]_{\frac{\sqrt{2}}{2}}^1 \\
&= \left(\ln(1) - \frac{1}{2}\cdot 1^2\right) - \left(\ln\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)^2\right) \\
&= -\frac{1}{4} - \ln\left(\frac{\sqrt{2}}{2}\right).
\end{aligned}$$

Problem 2. Find the area of the region bounded by $y = \sin^2(x)$ and $y = \sin^3(x)$, for $0 \le x \le \pi$.

Let us determine the points of intersection of $f(x) = \sin^2(x)$ and $g(x) = \sin^3(x)$. We have

$$\sin^2(x) = \sin^3(x) \Leftrightarrow 0 = \sin^3(x) - \sin^2(x) \quad \Leftrightarrow \quad 0 = \sin^2(x)(\sin(x) - 1),$$

which implies that

$$\sin^2(x) = 0 \quad \Leftrightarrow \quad x = 0,\ \pi \qquad \text{OR} \qquad \sin(x) = 1 \quad \Leftrightarrow \quad x = \pi/2.$$

This means we are to consider the subintervals $[0, \pi/2]$ and $[\pi/2, \pi]$, and we need to determine which of the functions $f(x) = \sin^2(x)$ and $g(x) = \sin^3(x)$ is "bigger" in each of these subintervals. We will pick a sample value in each interval and evaluate the functions at these values.

For the subinterval $[0, \pi/2]$, we will pick $x = \pi/4$. We have

$$f(\pi/4) = \sin^2(\pi/4) = (\sqrt{2}/2)^2 = 1/2$$

and

$$g(\pi/4) = \sin^3(\pi/4) = (\sqrt{2}/2)^3 = \frac{2\sqrt{2}}{8} = \frac{\sqrt{2}}{4} \approx 0.35.$$

Therefore, $\sin^2(x)$ is the "top" function over the interval $[0, \pi/2]$. Similarly, for the subinterval $[\pi/2, \pi]$, we can choose $x = (3\pi)/4$ and we can show that $\sin^2((3\pi)/4) > \sin^3((3\pi)/4)$, and hence, $\sin^2(x)$ is also the top function over the interval $[\pi/2, \pi]$.

We have determined that $\sin^2(x) \ge \sin^3(x)$ over the entire interval $[0, \pi]$. Then the area of the region is

$$\begin{aligned}
\int_0^\pi |\sin^2(x) - \sin^3(x)|\, \mathrm{dx} &= \int_0^\pi (\sin^2(x) - \sin^3(x))\, \mathrm{dx} \\
&= \int_0^\pi \sin^2(x)\, \mathrm{dx} - \int_0^\pi \sin^3(x)\, \mathrm{dx} \\
&= \int_0^\pi \frac{1}{2}(1 - \cos(2x))\, \mathrm{dx} - \int_0^\pi \sin^2(x)\sin(x)\, \mathrm{dx} \\
&= \int_0^\pi \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right) \mathrm{dx} - \int_0^\pi (1 - \cos^2(x))\sin(x)\, \mathrm{dx} \\
&\text{Let } u = \cos(x) \Rightarrow -\mathrm{du} = \sin(x)\, \mathrm{dx}, \\
&\text{When } x = 0 \Rightarrow u = \cos(0) = 1, \text{ and when } x = \pi \Rightarrow u = \cos(\pi) = -1 \\
&= \int_0^\pi \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right) \mathrm{dx} - \int_1^{-1} (1 - u^2)\, (\text{-du}) \\
&= \int_0^\pi \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right) \mathrm{dx} + 2\int_0^1 (u^2 - 1)\, \mathrm{du} \\
&= \left(\frac{1}{2}x - \frac{1}{4}\sin(2x)\right)\Big]_0^\pi + 2\left(\frac{1}{3}u^3 - u\right)\Big]_0^1 \\
&= \left(\frac{1}{2}\pi - \frac{1}{4}\sin(2\pi)\right) - \left(0 - \frac{1}{4}\sin(0)\right) + 2\Big[\left(\frac{1}{3}\cdot 1^3 - 1\right) - (0 - 0)\Big] \\
&= \left(\frac{1}{2}\pi - 0\right) - (0 - 0) - \frac{4}{3} \\
&= \frac{1}{2}\pi - \frac{4}{3}.
\end{aligned}$$

Section 7.3: Trigonometric Substitution

You may use the table below to help you compute integrals that involve expressions shown in the first column of the table. This table will be provided on future exams and quizzes.

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Problem 3. Evaluate the following integrals using a trigonometric substitution.

(a) $\displaystyle\int \frac{dx}{\sqrt{x^2-1}}$, (b) $\displaystyle\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{3/2}}\,dx$, (c) $\displaystyle\int \frac{dx}{\sqrt{x^2+2x+5}}\,dx$.

(a) By the table of trigonometric substitutions, we let

$$x = \sec(\theta) \quad \Rightarrow dx = \sec(\theta)\tan(\theta)\,d\theta.$$

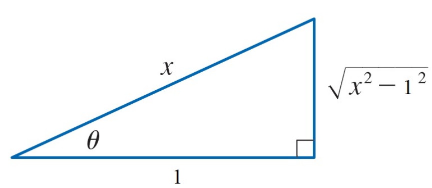
Then by using the associated trigonometric identity in the table, we have

$$\sqrt{x^2-1} = \sqrt{\sec^2(\theta)-1} = \sqrt{\tan^2(\theta)} = \tan(\theta).$$

In order to compute the integral, we will need the following antiderivative formula (this would be provided on quizzes/exams): $\int \sec(\theta)\,d\theta = \ln(|\sec(\theta)+\tan(\theta)|) + C$. We have

$$\int \frac{dx}{\sqrt{x^2-1}} = \int \frac{\sec(\theta)\tan(\theta)}{\tan(\theta)}\,d\theta = \int \sec(\theta)\,d\theta = \ln(|\sec(\theta)+\tan(\theta)|) + C.$$

Using our substitution $x = \sec(\theta)$, we draw the triangle below to help us finish computing the integral. The triangle tells us that $\tan(\theta) = \sqrt{x^2-1}$. Then



$$\sec\theta = x = \frac{x}{1}$$

$$\int \frac{dx}{\sqrt{x^2-1}} = \ln(|\sec(\theta)+\tan(\theta)|) + C = \ln(|x+\sqrt{x^2-1}|) + C.$$

(b) First note that

$$(4x^2+9)^{3/2} = \sqrt{(2x)^2+3^2}^{\,3},$$

so a trigonometric substitution can be applied. By the table of trigonometric substitutions, we let $2x = 3\tan(\theta)$, which is equivalent to

$$x = \frac{3}{2}\tan(\theta) \quad \Rightarrow \quad \mathrm{dx} = \frac{3}{2}\sec^2(\theta)\,d\theta.$$

Then by using the associated trigonometric identity in the table, we have

$$\sqrt{(2x)^2+3^2} = \sqrt{(3\tan(\theta))^2+3^2} = \sqrt{9\tan^2(\theta)+9} = 3\sqrt{\tan^2(\theta)+1} = 3\sqrt{\sec^2(\theta)} = 3\sec(\theta).$$

We also have to substitute the limits of integration. We have

$$0 = x = \frac{3}{2}\tan(\theta) \quad \Rightarrow \quad \theta = 0, \quad \text{and when} \quad \frac{3\sqrt{3}}{2} = x = \frac{3}{2}\tan(\theta) \Rightarrow \tan(\theta) = \sqrt{3} \quad \Rightarrow \quad \theta = \frac{\pi}{3}.$$

Then

$$\begin{aligned}
\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{3/2}}\,\mathrm{dx} = \int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{\sqrt{(2x)^2+3^2}^{\,3}}\,\mathrm{dx} &= \int_0^{\pi/3} \frac{(\frac{3}{2}\tan(\theta))^3}{(3\sec(\theta))^3}\frac{3}{2}\sec^2(\theta)\,d\theta \\
&= \int_0^{\pi/3} \frac{\frac{27}{8}\tan^3(\theta)}{27\sec^3(\theta)}\frac{3}{2}\sec^2(\theta)\,d\theta \\
&= \frac{3}{16}\int_0^{\pi/3} \frac{\tan^3(\theta)}{\sec(\theta)}\,d\theta \\
&= \frac{3}{16}\int_0^{\pi/3} \frac{\frac{\sin^3(\theta)}{\cos^3(\theta)}}{\frac{1}{\cos(\theta)}}\,d\theta \\
&= \frac{3}{16}\int_0^{\pi/3} \frac{\sin^3(\theta)}{\cos^2(\theta)}\,d\theta \quad \text{this is an integral from Section 7.2} \\
&= \frac{3}{16}\int_0^{\pi/3} \frac{\sin^2(\theta)}{\cos^2(\theta)}\sin(\theta)\,d\theta \\
&= \frac{3}{16}\int_0^{\pi/3} \frac{1-\cos^2(\theta)}{\cos^2(\theta)}\sin(\theta)\,d\theta
\end{aligned}$$

Let $u = \cos(x) \Rightarrow -\mathrm{du} = \sin(x)\,\mathrm{dx}$,

When $\theta = 0 \Rightarrow u = \cos(0) = 1$, and when $\theta = \pi/3 \Rightarrow u = \cos(\pi/3) = 1/2$

$$\begin{aligned}
&= \frac{3}{16}\int_1^{1/2} \frac{1-u^2}{u^2}\,(\text{-du}) \\
&= \frac{3}{16}\int_{1/2}^{1} (u^{-2}-1)\,\text{-du} \\
&= \frac{3}{16}\left(-u^{-1}-u\right)\Big]_{1/2}^{1} \\
&= \frac{3}{16}\left[(-1-1)-(-2-\frac{1}{2})\right] = \frac{3}{32}.
\end{aligned}$$

Note that we did not have to set up a triangle for this integral because it is a definite integral.

(c) In order to determine which trigonometric substitution we will need to apply, we will have to "complete the square." Note that

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2.$$

Then

$$x^2 + 2x + 5 = x^2 + 2x + 0 + 5 = x^2 + 2x + (1 - 1) + 5 = (x^2 + 2x + 1) + 4 = (x + 1)^2 + 2^2.$$

Therefore, we may apply the trigonometric substitution

$$x + 1 = 2\tan(\theta) \quad \Rightarrow \quad \mathrm{dx} = 2\sec^2(\theta)\,\mathrm{d}\theta.$$

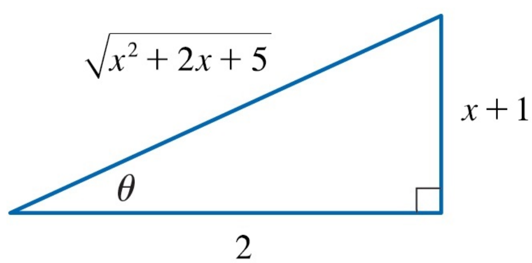
By using the associated trigonometric identity in the table, we have

$$\sqrt{(x+1)^2 + 2^2} = \sqrt{(2\tan(\theta))^2 + 2^2} = 2\sqrt{\tan^2(\theta) + 1} = 2\sqrt{\sec^2(\theta)} = 2\sec(\theta).$$

Then

$$\begin{aligned}\int \frac{\mathrm{dx}}{\sqrt{x^2 + 2x + 5}}\,\mathrm{dx} = \int \frac{\mathrm{dx}}{\sqrt{(x+1)^2 + 2^2}}\,\mathrm{dx} &= \int \frac{2\sec^2(\theta)}{2\sec(\theta)}\,\mathrm{d}\theta \\ &= \int \sec(\theta)\,\mathrm{d}\theta \\ &= \ln(|\sec(\theta) + \tan(\theta)|) + C.\end{aligned}$$

Using our substitution $x + 1 = 2\tan(\theta)$, we draw the triangle below to help us finish computing the integral. Then



$$\tan(\theta) = \frac{x+1}{2} \qquad \sec(\theta) = \frac{\sqrt{x^2 + 2x + 5}}{2}$$

$$\int \frac{\mathrm{dx}}{\sqrt{x^2 + 2x + 5}}\,\mathrm{dx} = \ln(|\sec(\theta) + \tan(\theta)|) + C = \ln\left(\left|\frac{\sqrt{x^2 + 2x + 5}}{2} + \frac{x+1}{2}\right|\right) + C.$$