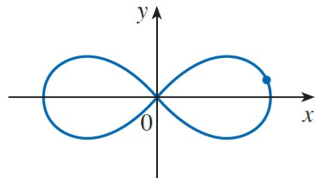
Section 3.5: Implicit Differentiation

Problem 1. The curve of the equation $2(x^2+y^2)^2 = 25(x^2-y^2)$ is called a *lemniscate*.



Find the points on the lemniscate where the tangent line is horizontal.

$$\begin{aligned}
&2(x^2+y^2)^2 = 25(x^2-y^2)\\
\Rightarrow\quad & 4(x^2+y^2)(2x+2y\,y') = 25(2x-2y\,y')\\
\Rightarrow\quad & 4(x+y\,y')(x^2+y^2) = 25(x-y\,y')\\
\Rightarrow\quad & 4y\,y'(x^2+y^2)+25y\,y' = 25x-4x(x^2+y^2)\\
\Rightarrow\quad & y' = \frac{25x-4x(x^2+y^2)}{25y+4y(x^2+y^2)}.
\end{aligned}$$

A tangent to the lemniscate will be horizontal if $y'=0 \quad\Rightarrow\quad 25x-4x(x^2+y^2)=0 \quad\Rightarrow$ $x[25-4(x^2+y^2)]=0 \quad\Rightarrow\quad x^2+y^2=\frac{25}{4}$ **(1)**.

Note that when x is 0, y is also 0, and there is no horizontal tangent at the origin.

Substituting $\frac{25}{4}$ for x^2+y^2 in the equation of the lemniscate, $2(x^2+y^2)^2=25(x^2-y^2)$, we get $x^2-y^2=\frac{25}{8}$ **(2)**.

Solving **(1)** and **(2)**, we have $x^2=\frac{75}{16}$ and $y^2=\frac{25}{16}$, so the four points are $\left(\pm\frac{5\sqrt{3}}{4},\pm\frac{5}{4}\right)$.

Problem 2. Use implicit differentiation to find an equation of the tangent line to the curve and the given point.

(a) $ye^{\sin(x)} = x\cos(y)$, $(0,0)$ (b) $\tan(x+y)+\sec(x-y)=2$ $(\pi/8,\pi/8)$.

HINT: After you apply implicit differentiation, plug in the x and y-coordinates of the given point FIRST, then solve for $y'=dy/dx$ to find the slope of the tangent line.

(a)

$$ye^{\sin x} = x\cos y \quad\Rightarrow\quad ye^{\sin x}\cdot\cos x + e^{\sin x}\cdot y' = x(-\sin y\cdot y')+\cos y\cdot 1$$

$$\Rightarrow\quad y'\,e^{\sin x}+y'\,x\sin y = \cos y - y\cos x\,e^{\sin x}$$

By plugging in $x = y = 0$ right away, we have

$$y' e^{\sin(0)} + y'(0)(\sin(0)) = \cos(0) - (0)\cos(0)e^{\sin(0)} \quad \Rightarrow \quad y' = \cos(0) = 1.$$

Then the slope of the tangent line is 1, so the equation of the line is $y = x$.

(b)

$$\tan(x+y) + \sec(x-y) = 2 \quad \Rightarrow \quad \sec^2(x+y)\cdot(1+y') + \sec(x-y)\tan(x-y)\cdot(1-y') = 0$$

By plugging in $x = y = \pi/8$ right away, we have

$$\sec^2(\pi/4)(1+y') + \sec(0)\tan(0)(1-y') = 0 \quad \Rightarrow \quad \left(\frac{2}{\sqrt{2}}\right)^2 (1+y') + (1)(0)(1-y') = 0$$
$$\Rightarrow \quad 2(1+y') = 0 \quad \Rightarrow \quad y' = -1$$

Then the slope of the tangent line is -1, so the equation of the line is $y - \pi/8 = -(x - \pi/8)$, or equivalently, $y = -x + \pi/4$.

Section 3.6: Derivatives of Logarithmic & Inverse Trigonometric Functions

Problem 3. Use properties of logarithms to expand the expression, then take the derivative.

(a) $\dfrac{d}{dt}\ln\left(\dfrac{t(t^2+1)^4}{\sqrt[3]{2t-1}}\right)$ (b) $\dfrac{d}{dx}\ln\left(\dfrac{e^{-x}\cos^2(x)}{x^2+x+1}\right)$

(a) Using properties of logarithms to fully expand the expression, we have

$$\ln\left(\frac{t(t^2+1)^4}{\sqrt[3]{2t-1}}\right) = \ln(t(t^2+1)^4) - \ln(\sqrt[3]{2t-1}) = \ln(t) + \ln((t^2+1)^4) - \ln((2t-1)^{1/3})$$
$$= \ln(t) + 4\ln(t^2+1) - \frac{1}{3}\ln(2t-1)$$

Then

$$\frac{d}{dt}\ln\left(\frac{t(t^2+1)^4}{\sqrt[3]{2t-1}}\right) = \frac{d}{dt}\ln(t) + 4\frac{d}{dt}\ln(t^2+1) - \frac{1}{3}\frac{d}{dt}\ln(2t-1)$$
$$= \frac{1}{t} + 4\frac{1}{t^2+1}(2t) - \frac{1}{3}\frac{1}{2t-1}(2)$$
$$= \frac{1}{t} + \frac{8t}{t^2+1} - \frac{2}{3(2t-1)}.$$

(a) Using properties of logarithms to fully expand the expression, we have

$$\ln\left(\frac{e^{-x}\cos^2(x)}{x^2+x+1}\right) = \ln(e^{-x}\cos^2(x)) - \ln(x^2+x+1) = \ln(e^{-x}) + \ln(\cos^2(x)) - \ln(x^2+x+1)$$
$$= -x + 2\ln(\cos(x)) - \ln(x^2+x+1)$$

Then

$$\frac{d}{dx}\ln\left(\frac{e^{-x}\cos^2(x)}{x^2+x+1}\right) = \frac{d}{dx}(-x) + 2\frac{d}{dx}\ln(\cos(x)) - \frac{d}{dx}\ln(x^2+x+1)$$
$$= -1 + \frac{2}{\cos(x)}(-\sin(x)) - \frac{1}{x^2+x+1}(2x+1)$$
$$= -1 - 2\tan(x) - \frac{2x+1}{x^2+x+1}.$$

Section 3.8: Exponential Growth/Decay

Problem 4. A culture of bacterium *Salmonella enteritidis* initially contails 50 cells. When introduced into a nutrient broth, the culture grows at a rate proportional to its size. After 1.5 hours, the population has increased to 975.

(a) Find an expression for the number of bacteria after t hours.
(b) Find the number of bacteria after 3 hours (round answer to nearest integer).
(c) Find the rate of growth after 3 hours (round answer to nearest integer).
(d) After how many hours will the population reach $250,000$?

(a) Let P(t)= number of bacteria in the colony after t hours.

Then $P(t) = P(0)e^{kt} = 50e^{kt}$. Now $P(1.5) = 50e^{k(1.5)} = 975 \quad \Rightarrow \quad e^{1.5k} = \frac{975}{50}$

$\Rightarrow\ 1.5k = \ln 19.5 \quad \Rightarrow \quad k = \frac{1}{1.5}\ln 19.5 \approx 1.9803$. So $P(t) \approx 50e^{1.9803t}$ cells.

(b) Using 1.9803 for k, we get $P(3) = 50e^{1.9803(3)} = 19{,}013.85 \approx 19{,}014$ cells.

(c) $\frac{dP}{dt} = kP \quad \Rightarrow \quad P'(3) = k \cdot P(3) = 1.9803 \cdot 19{,}014$ [from parts (a) and (b)] $= 37{,}653.4 \approx 37{,}653$ cells/h

(d) $P(t) = 50e^{1.9803t} = 250{,}000 \quad \Rightarrow \quad e^{1.9803t} = \frac{250{,}000}{50} \quad \Rightarrow \quad e^{1.9803t} = 5000 \quad \Rightarrow \quad 1.9803t = \ln 5000$

$\Rightarrow t = \frac{\ln 5000}{1.9803} \approx 4.30$ h

Problem 5. A bacteria culture grows with a constant relative growth rate. The bacteria count was 400 after 2 hours and $25,600$ after 6 hours.

(a) What is the relative growth rate? Express your answer as a percentage.
(b) Find the initial size of the culture (round answer to nearest integer).
(c) Find an expression for the number of bacteria after t hours.
(d) Find the number of bacteria after 4.5 hours (round answer to nearest integer).
(e) Find the rate of growth after 4.5 hours (round answer to nearest integer).
(f) When will the population reach $50,000$?

(a) $y(t) = y(0)e^{kt} \quad \Rightarrow \quad y(2) = y(0)e^{2k} = 400$ and $y(6) = y(0)e^{6k} = 25{,}600$.

Dividing these equations, we get

$e^{6k}/e^{2k} = 25{,}600/400 \quad \Rightarrow \quad e^{4k} = 64 \quad \Rightarrow \quad 4k = \ln 2^6 = 6\ln 2$

$\Rightarrow \quad k = \frac{3}{2}\ln 2 \approx 1.0397$, about 104% per hour.

(b) $400 = y(0)e^{2k} \quad \Rightarrow \quad y(0) = 400/e^{2k} \quad \Rightarrow \quad y(0) = 400/e^{3\ln 2} = 400/\left(e^{\ln 2}\right)^3 = 400/2^3 = 50.$

(c) $y(t) = y(0)e^{kt} = 50e^{(3/2)(\ln 2)t} = 50(e^{\ln 2})^{(3/2)t} \quad \Rightarrow \quad y(t) = 50(2)^{1.5t}$

(d) $y(4.5) = 50(2)^{1.5(4.5)} = 50(2)^{6.75} \approx 5382$ bacteria

(e) $\dfrac{dy}{dt} = ky = \left(\dfrac{3}{2}\ln 2\right)(50(2)^{6.75})$ [from parts (a) and (b)] ≈ 5596 bacteria/h

(f) $y(t) = 50{,}000 \quad \Rightarrow \quad 50{,}000 = 50(2)^{1.5t} \quad \Rightarrow \quad 1000 = (2)^{1.5t} \quad \Rightarrow \quad \ln 1000 = 1.5t\ln 2$

$\Rightarrow t = \dfrac{\ln 1000}{1.5\ln 2} \approx 6.64$ h