Section 13.1: Vector Functions & Space Curves

Problem 1. Find the domain of the vector function

$$\mathbf{r}(t) = \langle \ln(t+1), \frac{t}{\sqrt{9-t^2}}, 2^t \rangle.$$

Let $f(t) = \ln(t+1)$, $g(t) = \frac{t}{\sqrt{9-t^2}}$, and $h(t) = 2^t$. Since

$$\text{dom}(\mathbf{r}) = \text{dom}(f) \cap \text{dom}(g) \cap \text{dom}(h),$$

we will find the domains of each of f, g, and h.

Since $\text{dom}(\ln(x)) = (0, \infty)$, we have

$$\text{dom}(f) = \{t | t+1 > 0\} = \{t | t > -1\} = (-1, \infty).$$

Since $\text{dom}(\sqrt{x}) = [0, \infty)$ and $\text{dom}(1/x) = (-\infty, 0) \cup (0, \infty)$, we have

$$\text{dom}(g) = \left\{t | 9 - t^2 > 0\right\} = \{t | (3+t)(3-t) > 0\} = \{t | -3 < t < 3\} = (-3, 3).$$

Since $h(t) = 2^t$ is an exponential function, its domain is $\text{dom}(h) = \mathbb{R} = (-\infty, \infty)$.

Combining the above we obtain

$$\text{dom}(\mathbf{r}) = \text{dom}(f) \cap \text{dom}(g) \cap \text{dom}(h) = (-1, \infty) \cap (-3, 3) \cap (-\infty, \infty) = (-1, 3).$$

Problem 2. If

$$\mathbf{r}(t) = e^{-3t}\mathbf{i} + \frac{t^2}{\sin^2(t)}\mathbf{j} + \cos(2t)\mathbf{k},$$

find $\lim_{t \to 0} \mathbf{r}(t)$.

Let

$$f(t) = e^{-3t}, \qquad g(t) = \frac{t^2}{\sin^2(t)}, \qquad h(t) = \cos(2t).$$

Then since

$$\lim_{t \to 0} f(t) = \lim_{t \to 0} e^{-3t} = e^0 = 1,$$

$$\lim_{t \to 0} g(t) = \lim_{t \to 0} \frac{t^2}{\sin^2(t)} = \lim_{t \to 0} \frac{2t}{2\sin(t)\cos(t)} = \left(\lim_{t \to 0} \frac{t}{\sin(t)}\right)\left(\lim_{t \to 0} \frac{1}{\cos(t)}\right) = \left(\lim_{t \to 0} \frac{1}{\cos(t)}\right)\left(\frac{1}{1}\right) = \frac{1}{1} \cdot 1 = 1,$$

and

$$\lim_{t \to 0} h(t) = \lim_{t \to 0} \cos(2t) = \cos(0) = 1,$$

we have

$$\lim_{t \to 0} \mathbf{r}(t) = \lim_{t \to 0} \langle f(t), g(t), h(t) \rangle = \langle \lim_{t \to 0} f(t), \lim_{t \to 0} g(t), \lim_{t \to 0} h(t) \rangle = \langle 1, 1, 1 \rangle.$$

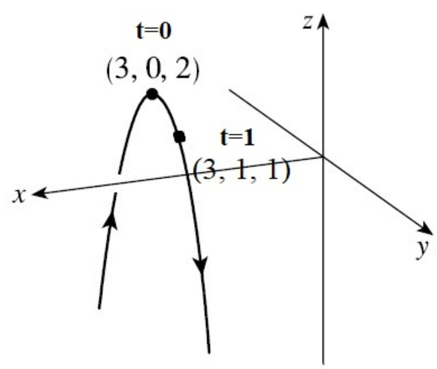
Problem 3. Sketch the curve with the given vector equation. Indicate with an arrow the direction in which t increases.

(a) $\mathbf{r}(t) = \langle 3, t, 2 - t^2 \rangle$, (b) $\mathbf{r}(t) = 2\cos(t)\mathbf{i} + 2\sin(t)\mathbf{j} + \mathbf{k}$, (c) $\mathbf{r}(t) = \cos(t)\mathbf{i} - \cos(t)\mathbf{j} + \sin(t)\mathbf{k}$.

(a) The corresponding parametric equations are $x = 3$, $y = t$, and $z = 2 - t^2$. By substituting $y = t$ into the z equation we obtain

$$z = 2 - y^2.$$

Since $x = 3$, the curve is the parabola $z = 2 - y^2$ on the plane $x = 3$. The vertex up the parabola is the point $(3, 0, 2)$, which is obtainted by letting $t = 0$. When $t = 1$, we have the point $(3, 1, 1)$.



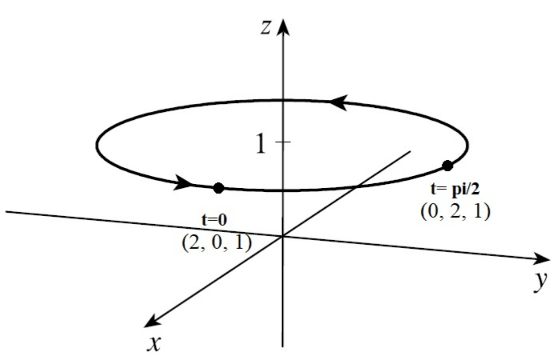
(b) The corresponding parametric equations are $x = 2\cos(t)$, $y = 2\sin(t)$, and $z = 1$. Note that

$$x^2 + y^2 = (2\cos(t))^2 + (2\sin(t))^2 = 2^2(\cos^2(t) + \sin^2(t)) = 2^2.$$

That is,

$$x^2 + y^2 = 2^2.$$

Since $z = 1$, the curve is a circle of radius 2 centered at $(0, 0, 1)$.



(c) The corresponding parametric equations are $x = \cos(t)$, $y = -\cos(t)$, and $z = \sin(t)$. Then

$$x^2 + z^2 = \cos^2(t) + \sin^2(t) = 1, \tag{1}$$

and similarly,

$$y^2 + z^2 = 1. \tag{2}$$

Then all points on the curve must satisfy (1) and (2), meaning that all points must lie on both of the surfaces of the circular cylinders of radius one along the y and x axes, i.e., the curve is contained in the intersection of these two cylinder. Furthermore, since $y = -x$, the curve also lies on the plane $y = -x$.

Section 13.2: Derivatives & Integrals of Vector Functions

Problem 4. Find the derivative of the vector function.

$$\mathbf{r}(t) = \langle t\,\sin(t), \sqrt{t^2-1}, 2^t\cos(2t)\rangle.$$

We have

$$f(t) = t\,\sin(t), \qquad g(t) = \sqrt{t^2-1}, \qquad h(t) = 2^t\cos(2t).$$

Since

$$f'(t) = \sin(t) + t\cos(t),$$
$$g'(t) = \frac{2t}{2\sqrt{t^2-1}} = \frac{t}{\sqrt{t^2-1}}.$$

and

$$h'(t) = \ln(2)\,2^t\cos(2t) - 2^{t+1}\sin(2t),$$

we have

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle = \langle 2^t\cos(2t),\ \sin(t)+t\cos(t),\ \ln(2)\,2^t\cos(2t) - 2^{t+1}\sin(2t)\rangle.$$

Problem 5. Find parametric equations for the tangent line to the curve with parametric equations

$$x = t^2+1, \qquad y = 4\sqrt{t}, \qquad z = e^{t^2-t}$$

at the point $(2,4,1)$.

Let us begin by finding the tangent vector function $\mathbf{r}'(t)$ of the vector function $\mathbf{r}(t) = \langle t^2+1, 4\sqrt{t}, e^{t^2-t}\rangle$. We have

$$\mathbf{r}'(t) = \langle 2t,\ 2/\sqrt{t},\ (2t-1)e^{t^2-t}\rangle.$$

Note that all three equations

$$2 = t^2+1, \quad 4 = 4\sqrt{t}, \quad \text{and} \quad 1 = e^{t^2-t}$$

are satisfied when $t = 1$. Thus, the direction vector of the tangent line is

$$\mathbf{r}'(1) = \langle 2, 2\sqrt{1}, (2-1)e^0\rangle = \langle 2,2,1\rangle.$$

Then the parametric equations of the tangent line are

$$x = 2+2t, \quad y = 4+2t, \quad z = 1+t.$$

Problem 6. If $\mathbf{r}(t) = t\cos(t^2)\,\mathbf{i} + \frac{1}{t}\,\mathbf{j} + 2t^{3/2}\,\mathbf{k}$, find $\int \mathbf{r}(t)\,\mathrm{dt}$.

Let $f(t) = t\cos(t^2)$, $g(t) = 1/t$, and $h(t) = 2t^{3/2}$. We will find the indefinite integrals of f, g, and h with respect to t.

Let $u = t^2$. Then $\mathrm{du} = 2t\,\mathrm{dt}$, or equivalently, $\frac{1}{2}\mathrm{du} = t\,\mathrm{dt}$. Then

$$\int f(x)\,\mathrm{dt} = \int t\cos(t^2)\mathrm{dt} = \frac{1}{2}\int\cos(u)\mathrm{du} = \frac{1}{2}\sin(u) + C_1 = \frac{1}{2}\sin(t^2) + C_1,$$

where C_1 is a constant.

For the funcion g, we have

$$\int g(t)\,\mathrm{dt} = \int \frac{1}{t}\mathrm{dt} = \ln(|t|) + C_2,$$

where C_2 is a constant.

Finally, for h we have

$$\int h(x)\,\mathrm{dt} = \int 2t^{3/2}\mathrm{dt} = 2\int t^{3/2}\mathrm{dt} = 2\frac{2}{5}t^{5/2} + C_3 = \frac{4}{5}t^{5/2} + C_3.$$

Then

$$\begin{aligned}\int \mathbf{r}(t)\,\mathrm{dt} &= \left(\int f(x)\,\mathrm{dt}\right)\mathbf{i} + \left(\int g(x)\,\mathrm{dt}\right)\mathbf{j} + \left(\int h(x)\,\mathrm{dt}\right)\mathbf{k}\\ &= \left(\frac{1}{2}\sin(t^2) + C_1\right)\mathbf{i} + (\ln(|t|) + C_2)\,\mathbf{j} + \left(\frac{4}{5}t^{5/2} + C_3\right)\mathbf{k}\\ &= \left(\frac{1}{2}\sin(t^2)\right)\mathbf{i} + (\ln(|t|))\,\mathbf{j} + \left(\frac{4}{5}t^{5/2}\right)\mathbf{k} + \mathbf{C},\end{aligned}$$

where $\mathbf{C} = \langle C_1, C_2, C_3\rangle$.

Problem 7. If $\mathbf{u}(t) = \langle \sin(t), \cos(t), t\rangle$ and $\mathbf{v}(t) = \langle t, \cos(t), \sin(t)\rangle$ find

(a) $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)]$, (b) $\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)]$.

$$\begin{aligned}\frac{d}{dt}\left[\mathbf{u}(t)\cdot\mathbf{v}(t)\right] &= \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)\\ &= \langle \cos t, -\sin t, 1\rangle\cdot\langle t, \cos t, \sin t\rangle + \langle \sin t, \cos t, t\rangle\cdot\langle 1, -\sin t, \cos t\rangle\\ &= t\cos t - \cos t\,\sin t + \sin t + \sin t - \cos t\,\sin t + t\cos t\\ &= 2t\cos t + 2\sin t - 2\cos t\,\sin t\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\left[\mathbf{u}(t)\times\mathbf{v}(t)\right] &= \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)\\ &= \langle \cos t, -\sin t, 1\rangle\times\langle t, \cos t, \sin t\rangle + \langle \sin t, \cos t, t\rangle\times\langle 1, -\sin t, \cos t\rangle\\ &= \left\langle -\sin^2 t - \cos t, t - \cos t\,\sin t, \cos^2 t + t\sin t\right\rangle\\ &\qquad + \left\langle \cos^2 t + t\sin t, t - \cos t\,\sin t, -\sin^2 t - \cos t\right\rangle\\ &= \left\langle \cos^2 t - \sin^2 t - \cos t + t\sin t, 2t - 2\cos t\,\sin t, \cos^2 t - \sin^2 t - \cos t + t\sin t\right\rangle\end{aligned}$$