Section 5.5: The Substitution Rule

SOLUTIONS

Problem 1. Evaluate the indefinite integral.

(a) $\int \cos^3(\theta)\sin(\theta)\,d\theta$, (b) $\int \frac{\sin(2x)}{2+\cos^2(x)}\,dx$, (c) $\int x\sqrt{x+2}\,dx$.

(a) Let $u = \cos(\theta)$. Then $du = -\sin(\theta)\,d\theta$

$\Downarrow$

$-du = \sin(\theta)\,d\theta$.

Then we can rewrite

$$\int \cos^3(\theta)\sin(\theta)\,d\theta = \int u^3(-du) = -\int u^3\,du = -\frac{1}{4}u^4 + C = \boxed{-\frac{1}{4}\cos^4(\theta) + C}$$

(b) HINT: $\sin(2x) = 2\cos(x)\sin(x)$ ⇝ TRIGONOMETRIC DOUBLE-ANGLE FORMULA.

Then

$$\int \frac{\sin(2x)}{2+\cos^2(x)}\,dx = \int \frac{2\cos(x)\sin(x)}{2+\cos^2(x)}\,dx = -\int \frac{1}{u}\,du$$

$$= -\ln(|u|) + C$$

$$= \boxed{-\ln(|2+\cos^2(x)|) + C}$$

Let $u = 2 + \cos^2(x)$.

Then $du = -2\cos(x)\sin(x)\,dx$

$\Downarrow$

$-du = 2\cos(x)\sin(x)\,dx$

(c) $$\int x\sqrt{x+2}\,dx = \int (u-2)\sqrt{u}\,du = \int (u-2)u^{\frac{1}{2}}\,du$$

$$= \int \left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right)du = \frac{2}{5}u^{\frac{5}{2}} - \frac{4}{3}u^{\frac{3}{2}} + C$$

$$= \boxed{\frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C}$$

Let $u = x+2$.

Then $du = dx$.

How do we subs. x?

Well, $u = x+2$

$\Downarrow$

$x = u-2$.

CAREFUL!

$$\sqrt{x+2} = (x+2)^{\frac{1}{2}} \neq x^{\frac{1}{2}} + 2^{\frac{1}{2}} = \sqrt{x} + \sqrt{2}$$

Problem 2. Evaluate the definite integral.

(a) $\int_1^2 \frac{e^{1/x}}{x^2}\,dx$, (b) $\int_{-\pi}^{\pi} \sin^2(\theta)\cos(\theta)\,d\theta$, (c) $\int_{-\pi/2}^{\pi/2}\left(x^3 + \frac{1+x^2}{\sin(x)}\right) dx$.

(a) $$\int_1^2 \frac{e^{\frac{1}{x}}}{x^2}\,dx = -\int_1^{\frac{1}{2}} e^u\,du = \int_{\frac{1}{2}}^{1} e^u\,du = e - e^{\frac{1}{2}} = e - \sqrt{e}$$

$u = \frac{1}{x} = x^{-1}$

$du = -x^{-2}\,dx$

$-du = \frac{1}{x^2}\,dx$

When

$x=2 \rightsquigarrow u = \frac{1}{2}$

$x=1 \rightsquigarrow u = \frac{1}{1} = 1$

(b) Let $f(\theta) = \sin^2(\theta)\cos(\theta)$. Then $f(-\theta) = \sin^2(-\theta)\cos(-\theta)$

$= (-\sin(\theta))^2\cos(\theta)$

$= \sin^2(\theta)\cos(\theta)$

$= f(\theta)$.

$\Rightarrow f$ is even. Then

$u = \sin(\theta)$

$du = \cos(\theta)\,d\theta$

When

$x = \pi \rightsquigarrow u = \sin(\pi) = 0$

$x = 0 \rightsquigarrow u = \sin(0) = 0$

$$\int_{-\pi}^{\pi} \sin^2(\theta)\cos(\theta)\,d\theta = 2\int_0^{\pi} \sin^2(\theta)\cos(\theta)\,d\theta$$

$$= 2\int_0^0 u^2\,du = 0, \quad \text{since } \int_a^a f(x)\,dx = 0.$$

(c) Let $f(x) = x^3 + \frac{1+x^2}{\sin(x)}$. Then $f(-x) = (-x)^3 + \frac{1+(-x)^2}{\sin(-x)}$

$= -x^3 + \frac{1+x^2}{-\sin(x)}$

$= -\left(x^3 + \frac{1+x^2}{\sin(x)}\right) = -f(x)$

$\Rightarrow f$ is odd. Then

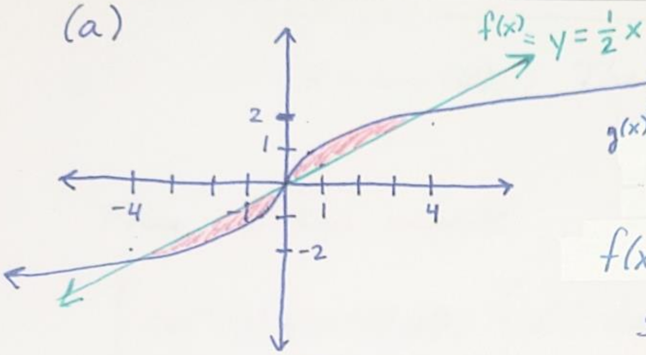
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(x^3 + \frac{1+x^2}{\sin(x)}\right) dx = 0.$$

Section 6.1: Areas Between Curves

Problem 3. Sketch the region enclosed by the given curves. Decide whether to integrate with respect to x or y and find the area of the region.

(a) $y = \sqrt[3]{2x}$, $y = \frac{1}{2}x$, (b) $4x + y^2 = 12$, $x = y$, (c) $y = (x-2)^2$, $y = x$, $x = 1$, $x = 3$.

(a)



$f(x) = \frac{1}{2}x \qquad g(x) = \sqrt[3]{2x}$

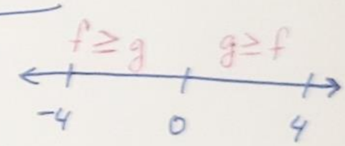
Solve $f(x) = g(x)$

$\frac{1}{2}x = \sqrt[3]{2x} \Leftrightarrow \frac{1}{8}x^3 = 2x$

$\Leftrightarrow x^3 = 16x$

$\Leftrightarrow x(x^2-16) = 0$

$\Leftrightarrow x = 0, \quad x = \pm 4.$



$$\int_{-4}^{4} |f(x) - g(x)|\, dx$$

$$= \int_{-4}^{4} \left| \tfrac{1}{2}x - \sqrt[3]{2x} \right| dx$$

$$= 2\int_{-4}^{4} \left(\sqrt[3]{2x} - \tfrac{1}{2}x\right) dx$$

$$= 2\left(\tfrac{3}{4}(2x)^{\frac{4}{3}} - \tfrac{1}{4}x^2\right)\Big]_0^4$$

$$= 2\left(\tfrac{3}{4}\cdot 8^{\frac{4}{3}} - 4 - (0-0)\right)$$

$$= 2(12-4) = 16$$

Note that

$f(-x) = \frac{1}{2}(-x) = -\frac{1}{2}x = -f(x)$

$g(-x) = \sqrt[3]{2(-x)} = \sqrt[3]{-1}\,\sqrt[3]{2x}$

$= -\sqrt[3]{2x} = -g(x)$

Then

$|f(-x) - g(-x)|$

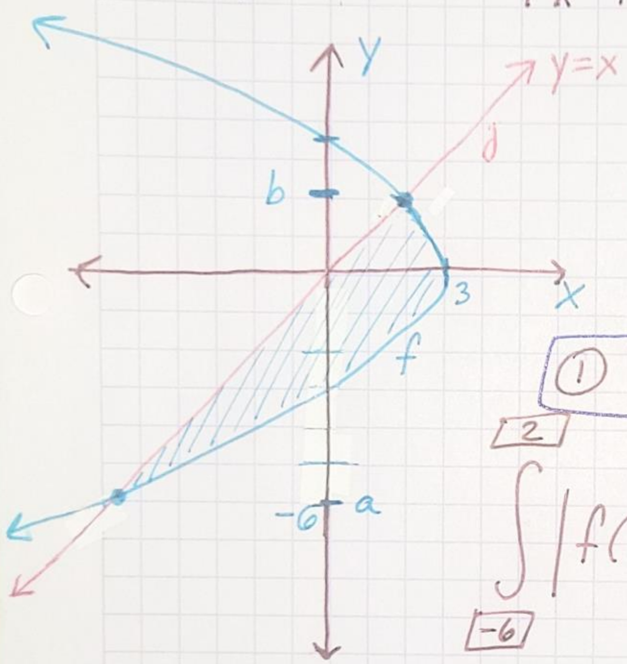
$= |f(x) - g(x)|$

$\Rightarrow |f(x) - g(x)|$ is even

(b)

$$4x + y^2 = 12, \qquad x = y$$

$$x = -\frac{1}{4}y^2 + 3 = f(y) \qquad x = y = g(y)$$



$\boxed{\text{① Integrate w.r.t. } y.}$

$$\int_{\boxed{-6}}^{\boxed{2}} |f(y) - g(y)|\, dy$$

$$= \int_{\boxed{-6}}^{\boxed{2}} \left| \left(-\frac{1}{4}y^2 + 3\right) - y \right| dy$$

$$F(y) = -\frac{1}{12}y^3 - \frac{y^2}{2} + 3y$$

$$F(2) = \frac{10}{3}$$

$$F(-6) = -18$$

$$= \int_{-6}^{2} \left(-\frac{1}{4}y^2 - y + 3\right) dy$$

$$= F(y)\Big]_{-6}^{2} = F(2) - F(-6) = \frac{10}{3} - (-18) = \frac{64}{3}$$

$f(y) = g(y)$

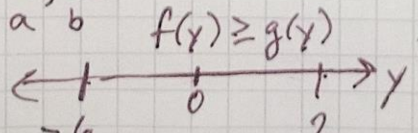
$-\frac{1}{4}y^2 + 3 = y$

$0 = \frac{1}{4}y^2 + y - 3$

$0 = y^2 + 4y - 12$

$0 = (y+6)(y-2)$

$\Rightarrow y = \underset{a}{-6}, \underset{b}{2}$



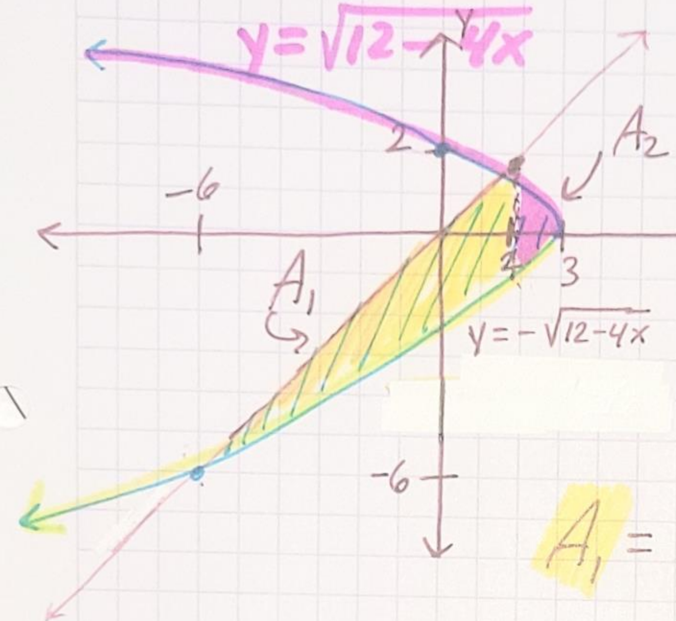
$f(y) - g(y) \geq 0$

$f(0) = 3$

$g(0) = 0$

$$4x + y^2 = 12, \qquad x = y$$

② Integrating w.r.t. x



$f(x) \qquad g(x)$

$$y = \pm\sqrt{12-4x}$$

$$\Rightarrow y \neq f(x)$$

y cannot be expressed as a function of x.

$$A_1 = \int_{-6}^{2} \left(x - \left(-\sqrt{12-4x}\right)\right) dx$$

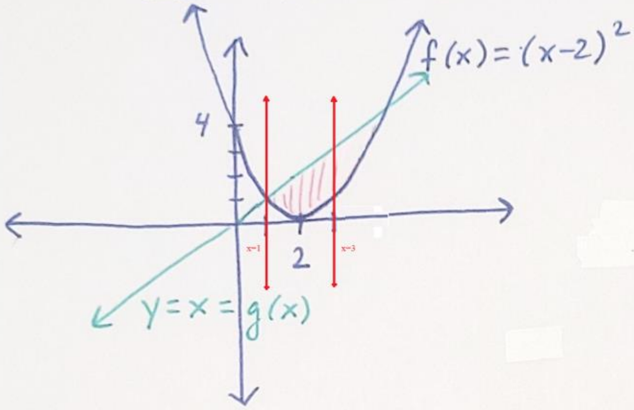
$$= \int_{-6}^{2} \left(x + \sqrt{12-4x}\right) dx$$

$$A_2 = 2\int_{2}^{3} \sqrt{12-4x}\, dx$$

$$\text{AREA} = A_1 + A_2 = \frac{64}{3}$$

(c) $y=(x-2)^2, \quad y=x, \quad x=1, \quad x=3$

Let $f(x)=(x-2)^2$ and $g(x)=x$.



$f(x)=g(x)$

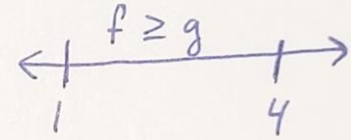
$(x-2)^2=x$

$x^2-4x+4=x$

$x^2-5x+4=0$

$(x-1)(x-4)=0$

$x=1,4$



$$\text{AREA} = \int_1^3 |f(x)-g(x)|\,dx$$

since $f \geq g$ on $[1,3]$

$$= \int_1^3 (f(x)-g(x))\,dx$$

$$= \int_1^3 \left((x-2)^2-x\right)dx$$

$$= \int_1^3 (x^2-5x+4)\,dx = F(3)-F(1) = \frac{35}{2}-0 = \frac{35}{2}$$

$F(x)=\frac{1}{3}x^3-\frac{5}{2}x^2+4x$

$F(3)=\frac{1}{3}\cdot 27-\frac{5}{2}\cdot 3+16=\frac{35}{2}$

$F(1)=1-5+4=0$