Section 11.2: Series

Problem 1. Determine whether the series is convergent or divergent by expressing its partial sum $s_n = \sum_{i=1}^{n} a_i$ as a telescoping sum. If it is convergent, find its sum.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$. **Hint:** Express $\frac{3}{i(i+3)} = \frac{A}{i} + \frac{B}{i+3}$ and find A and B by the Method of Partial Fractions.

(b) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$. **Hint:** $\ln\left(\frac{A}{B}\right) = \ln(A) - \ln(B)$.

(a) Let us compute the partial sum formula $s_n = \sum_{i=1}^{n} \frac{3}{i(i+3)}$ by expressing $\frac{3}{i(i+3)}$ as the sum of two fractions using the Method of Partial Fractions. We have

$$\frac{3}{i(i+3)} = \frac{A}{i} + \frac{B}{i+3} = \frac{A(i+3)+Bi}{i(i+3)} \Leftrightarrow 3 + 0i = 3 = 3A + (A+B)i$$

Then we have $3 = 3A$, which implies that $A = 1$ and $0 = A + B$, which implies that $B = -A = -1$. Then

$$\begin{aligned}
s_n &= \sum_{i=1}^{n} \frac{3}{i(i+3)} \\
&= \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+3}\right) \\
&= \left(\frac{1}{1} - \cancel{\frac{1}{4}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{5}}\right) + \left(\frac{1}{3} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{7}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{8}}\right) + \left(\cancel{\frac{1}{6}} - \cancel{\frac{1}{9}}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+3}\right) \\
&= 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+3} = \frac{11}{6} - \frac{1}{n+3}.
\end{aligned}$$

Then

$$\sum_{n=1}^{\infty} \frac{3}{n(n+3)} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{11}{6} - \frac{1}{n+3}\right) = \frac{11}{6} - 0 = \frac{11}{6}.$$

Therefore, the series is **convergent**.

(b) By using properties of logarithms, we have that the partial sum

$$\begin{aligned}
s_n = \sum_{i=1}^{n} \ln\left(\frac{i}{i+1}\right) &= \sum_{i=1}^{n} \left(\ln(i) - \ln(i+1)\right) \\
&= \left(\ln(1) - \cancel{\ln(2)}\right) + \left(\cancel{\ln(2)} - \cancel{\ln(3)}\right) + \left(\cancel{\ln(3)} - \cancel{\ln(4)}\right) + \cdots + \left(\cancel{\ln(n)} - \ln(n+1)\right) \\
&= \ln(1) - \ln(n+1) \\
&= -\ln(n+1).
\end{aligned}$$

Then

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \lim_{n\to\infty} s_n = \lim_{n\to\infty} -\ln(n+1) = -\infty.$$

Therefore, the series is **divergent**.

Problem 2. Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

(a) $2 + 0.5 + .125 + 0.03125 + \cdots,$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{2n}}{6^{n-1}}.$

(a) Note that the first few terms of the geometric series are $a + ar + ar^2 + \ldots$. By comparing this with $2 + 0.5 + .125 + 0.03125 + \cdots$, we see that

$$a = 2 \quad \Rightarrow \quad 2r = ar = 0.5 \quad \Rightarrow \quad r = \frac{0.5}{2} = \frac{1/2}{2} = \frac{1}{4}.$$

One can check that $0.125 = 2(1/4)^2$ and $0.03125 = 2(1/20)^3$. Then

$$2 + 0.5 + .125 + 0.03125 + \cdots = \sum_{n=1}^{\infty} 2\left(\frac{1}{4}\right)^{n-1} = \frac{2}{1-1/4} = \frac{2}{3/4} = \frac{8}{3},$$

which means that the series **converges**.

(b) Note that

$$\frac{e^{2n}}{6^{n-1}} = \frac{(e^2)^n}{6^{n-1}} = e^2\frac{(e^2)^{n-1}}{6^{n-1}} = e^2\left(\frac{e^2}{6}\right)^{n-1}.$$

Then

$$\sum_{n=1}^{\infty} \frac{e^{2n}}{6^{n-1}} = \sum_{n=1}^{\infty} e^2\left(\frac{e^2}{6}\right)^{n-1},$$

meaning that $a = e^2$ and $r = e^2/6 \approx (2.7)^2/6 = 7.29/6$. Then $|r| \geq 1$, which means that the series **diverges**.

Problem 3. Determine whether the series is convergent or divergent. If it is convergent, find its sum.

(a) $\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + \frac{1}{15} + \cdots$ (b) $\displaystyle\sum_{n=1}^{\infty}[(-0.2)^n + (0.6)^{n-1}]$ (c) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right)$ (d) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{5^n} + \frac{2}{n}\right)$

(a) $\displaystyle\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + \frac{1}{15} + \cdots = \sum_{n=1}^{\infty} \frac{1}{3n} = \frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n}$. This is a constant multiple of the divergent harmonic series, so it **diverges.**

(b) Note that

$$\lim_{n\to\infty}[(-0.2)^n + (0.6)^{n-1}] = 0 + 0 = 0,$$

so we cannot use the Divergence Test. Note that

$$\sum_{n=1}^{\infty}[(-0.2)^n + (0.6)^{n-1}] = \sum_{n=1}^{\infty}(-0.2)(-0.2)^{n-1} + \sum_{n=1}^{\infty}(0.6)^{n-1},$$

so both of the individual series that make up the entire series are geometric. Since $|-0.2| < 1$ and $|0.6| < 1$, both of the series converge, so

$$\sum_{n=1}^{\infty}[(-0.2)^n + (0.6)^{n-1}] = \sum_{n=1}^{\infty}(-0.2)(-0.2)^{n-1} + \sum_{n=1}^{\infty}(0.6)^{n-1} = \frac{-0.2}{1-(-0.2)} + \frac{1}{1-0.6} = -\frac{1}{6} + \frac{5}{2} = \frac{7}{3}.$$

(c) $\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right)$ **diverges** by the Test for Divergence since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right) = \ln\left(\lim_{n\to\infty} \frac{n^2+1}{2n^2+1}\right) = \ln \tfrac{1}{2} \neq 0.$$

(d) $\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n}\right)$ diverges because $\sum_{n=1}^{\infty} \frac{2}{n} = 2\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. (It is 2 times the Harmonic Series, which diverges).