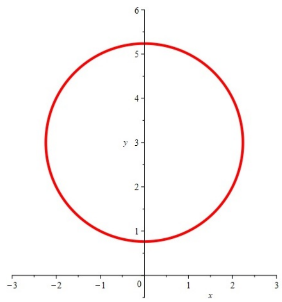
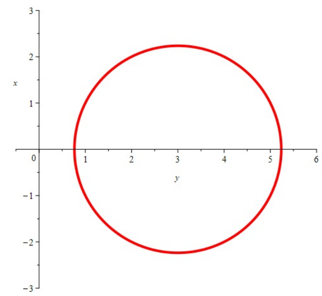
Section 1.1: Four Ways to Represent a Function

Problem 1. In each of the following equations, please determine whether y is a function of x and if x is a function of y. Please support your answers with an explanation.

(a) $x^2 + (y-3)^2 = 5$ (b) $x + (y-3)^3 = 5$

(a) The equation is that of a circle of radius $\sqrt{5}$ with center $(0, 3)$. By graphing the equation by treating x or y as the input variable, we obtain the graphs shown below.



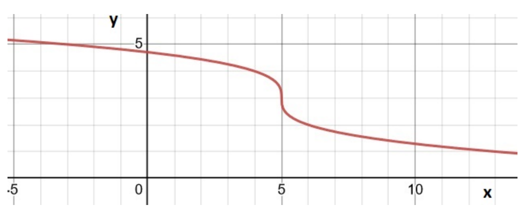


If we were to draw the vertical line $x = 1$ on the graph on the left, we see that it would cross the circle twice, meaning that it does not pass the Vertical Line Test (VLT). Therefore, y is **not** a function of x. If we were to draw the vertical line $y = 1$ on the graph on the right, we see that it would cross the circle twice, meaning that it does not pass the VLT. Therefore, x is **not** a function of y.

(b) By solving the equation for y, we obtain

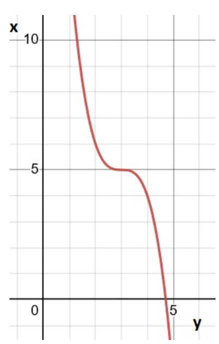
$$(y-3)^3 = 5 - x \quad \Rightarrow \quad y - 3 = \sqrt[3]{5-x} \quad \Rightarrow \quad y = \sqrt[3]{5-x} + 3.$$

The equation is a transformation of the function $y = \sqrt[3]{x}$. The graph is shown below. The graph above



passes the VLT. Then y is a function of x.

Solving the equation for x, we have $x = 5 - (y-3)^3$, which is a transformation of the function $x = y^3$. Its graph (shown below) passes the VLT. Therefore, x is a function of y.

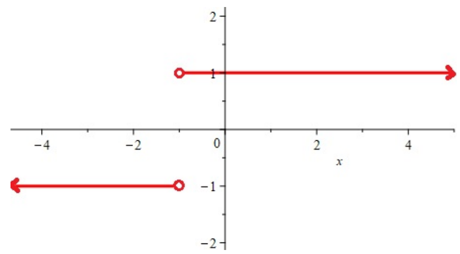


Problem 2. Sketch the graph of each of the following functions.

(a) $f(x) = \dfrac{|x+1|}{x+1}$ (b) $g(x) = ||x|-1|$

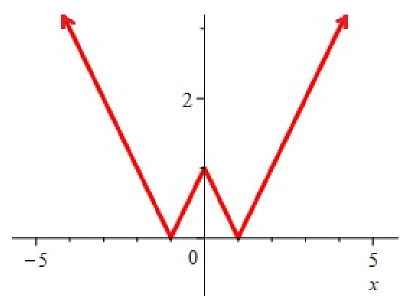
(a) Note that $x+1 \neq 0$, since $x=-1$ is not in the denominator of the function. We have

$$f(x) = \frac{|x+1|}{x+1} = \begin{cases} \dfrac{x+1}{x+1} & \text{if } x+1>0 \\ \\ \dfrac{-(x+1)}{x+1} & \text{if } x+1<0 \end{cases} = \begin{cases} 1 & \text{if } x>-1 \\ -1 & \text{if } x<-1. \end{cases}$$



(b) We have

$$g(x) = ||x|-1| = \begin{cases} |x|-1 & \text{if } |x|-1 \geq 0 \\ -(|x|-1) & \text{if } |x|-1<0 \end{cases} = \begin{cases} |x|-1 & \text{if } x \leq -1 \text{ OR } x \geq 1 \\ -|x|+1 & \text{if } -1<x<1 \end{cases}.$$



Problem 3. Find the domain of each of the following functions:

$$(a)\ f(x) = \frac{\sqrt{x^2-1}}{x-2}, \qquad (b)\ g(x) = \frac{1}{\sqrt[4]{x^2-5x}}.$$

Please state your answer in set or interval notation.

(a) We have

$$\begin{aligned} \text{dom}(f) &= \{x \mid x^2-1 \geq 0,\, x-2 \neq 0\} \\ &= \{x \mid x^2 \geq 1,\, x \neq 2\} \\ &= \{x \mid x \geq 1,\, x \leq -1,\, x \neq 2\} \\ &= (-\infty,-1) \cup (1,2) \cup (2,\infty). \end{aligned}$$

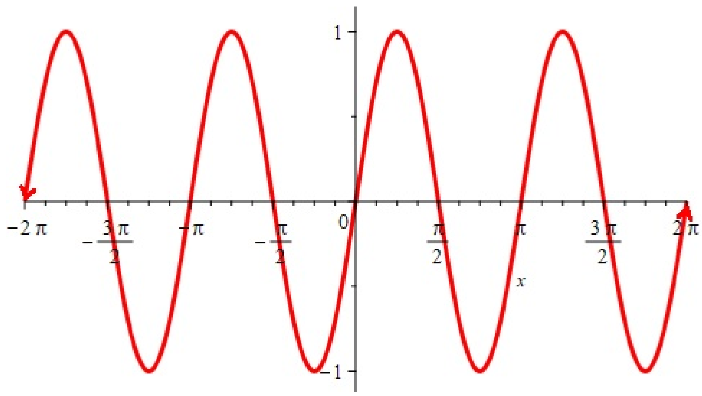
(b) We have

$$\begin{aligned}\text{dom}(g) &= \{x \mid x^2 - 5x > 0\}\\ &= \{x \mid x(x-5) > 0\}\\ &= \{x \mid x < 0,\ x > 5\}\\ &= (-\infty, 0) \cup (5, \infty).\end{aligned}$$

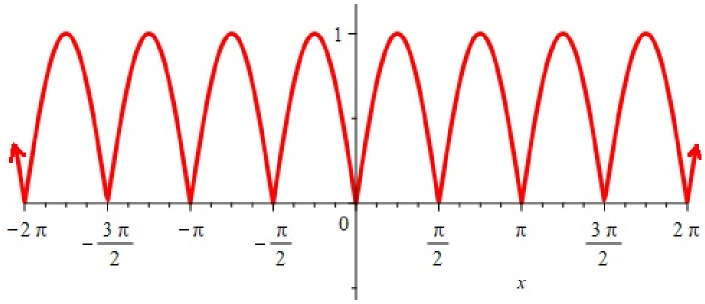
Section 1.3: New Functions from Old Functions

Problem 4. Use the graph of $f(x) = \sin(2x)$ to sketch the graph of $g(x) = |\sin(2x)|$.

The graph of $f(x) = \sin(2x)$ is a transformation of $y = \sin(x)$ (horizontal shrink by a factor of 2). Its graph is shown below.



To obtain the graph of $g(x) = |\sin(2x)|$, we take the graph of $f(x) = \sin(2x)$ and reflect all parts of the graph that lie below the x-axis over the x-axis. The graph of g is shown below.



Problem 5. Let $f(x) = \dfrac{1}{x-1}$ and $g(x) = \sqrt{x-1}$.

Find $f+g$, $f-g$, fg, and f/g and state each of their domains.

We have

$$(f+g)(x) = f(x) + g(x) = \frac{1}{x-1} + \sqrt{x-1},$$

$$(f-g)(x) = f(x) - g(x) = \frac{1}{x-1} - \sqrt{x-1},$$

and

$$(fg)(x) = f(x)\cdot g(x) = \frac{1}{x-1}\left(\sqrt{x-1}\right) = \frac{\sqrt{x-1}}{x-1}.$$

Since the domain of f is

$$\mathrm{dom}(f) = \{x \mid x \neq 1\} = (-\infty, 1) \cup (1, \infty),$$

and the domain of g is

$$\mathrm{dom}(g) = \{x \mid x \geq 1\} = [1, \infty),$$

the domains of $f+g$, $f-g$, and fg are

$$\mathrm{dom}(f+g) = \mathrm{dom}(f-g) = \mathrm{dom}(fg) = \mathrm{dom}(f) \cap \mathrm{dom}(g) = (1, \infty).$$

The function

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\frac{1}{x-1}}{\sqrt{x-1}} = \frac{1}{(x-1)\sqrt{x-1}} = \frac{1}{(x-1)^{3/2}}.$$

Since

$$\{x \mid g(x) \neq 0\} = \{x \mid x \neq 1\} = (-\infty, 1) \cup (1, \infty),$$

then

$$\mathrm{dom}\left(\frac{f}{g}\right) = \mathrm{dom}(f) \cap \mathrm{dom}(g) \cap \{x \mid g(x) \neq 0\} = (1, \infty).$$

Problem 6. Given $F(x) = \sqrt[3]{x^2+1} + 6$, find functions f, g, and h such that $F = f \circ g \circ h$.

We can let

$$h(x) = x^2 + 1, \qquad g(x) = \sqrt[3]{x}, \qquad , f(x) = x + 6.$$

Let us check that $F = f \circ g \circ h$. Since

$$(g \circ h)(x) = g(h(x)) = g(x^2+1) = \sqrt[3]{x^2+1},$$

then

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(\sqrt[3]{x^2+1}) = \sqrt[3]{x^2+1} + 6 = F(x).$$