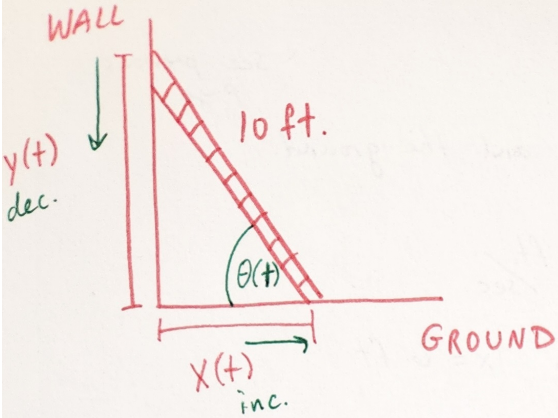
Section 3.9: Related Rates

Problem 1. A ladder 10 feet long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 4 feet per second, how fast is the angle between the ladder and the ground changing when the bottom of the ladder is 6 feet from the wall?



$\theta(t)$ = angle between ladder & ground after t sec.

GIVEN: $x'(t) = 4$ ft/sec.

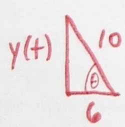
GOAL: Find $\theta'(t)$ when $x(t) = 6$ ft.

EQN. THAT RELATES KNOWNS/UNKNOWNS: $\cos\big(\theta(t)\big) = \dfrac{x(t)}{10}$

Apply implicit diff. w.r.t. t: $-\sin\big(\theta(t)\big)\theta'(t) = \dfrac{x'(t)}{10}$

$$\Rightarrow \theta'(t) = \frac{x'(t)}{\underbrace{-\sin(\theta(t))}_{?}\cdot 10}$$

When $x(t) = 6$, we have



$\Rightarrow y(t) = \sqrt{10^2 - 6^2} = 8$ ft.

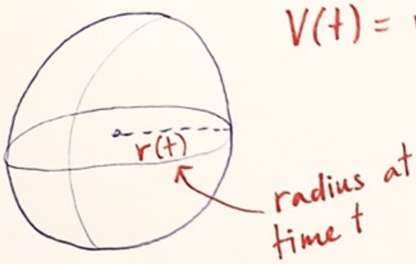
$\Rightarrow \sin(\theta(t)) = \dfrac{y(t)}{10} = \dfrac{8}{10}$ when $x(t) = 6$ ft.

Then $\theta'(t) = \dfrac{4}{-\frac{8}{10}\cdot 10} = -\dfrac{4}{8}$

$= -\dfrac{1}{2}$ radians per second.

Problem 2. Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the baloon increasing when the diameter is 50 cm?

$V(t)$ = volume at time t



GIVEN: $V'(t) = 100\ \text{cm}^3/\text{sec.}$

GOAL: Find $r'(t)$ when $r(t) = 25\text{cm}$

(if diameter $= 50$ $\Rightarrow$ radius $= 25$)

RELATING EQN.: $V(t) = \frac{4}{3}\pi r^3(t)$ (* not expected to know volume of sphere by memory *)

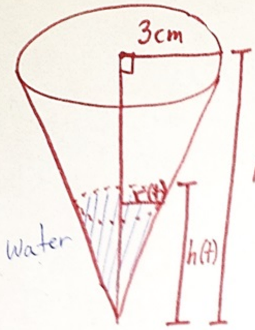
$$\Downarrow$$

$$V'(t) = \frac{4}{\cancel{3}}\pi \cancel{3} r^2(t) \cdot r'(t)$$

$$\Rightarrow 100 = 4\pi (25)^2 r'(t)$$

$$\Rightarrow r'(t) = \frac{100}{4\pi(25)^2} = \frac{1}{25\pi} \approx 0{,}0127\ \text{cm}/\text{sec.}$$

Problem 3. A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of 2 cm^3/s, how fast is the water level rising when the water is 5 cm deep?



$h(t)$ = height of water at t sec.

$r(t)$ = radius of the water surface at time t

$V(t)$ = volume of water at time t.

GIVEN: $V'(t) = 2\ cm^3/sec.$

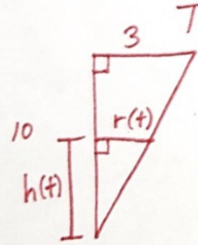
GOAL: Find $h'(t)$ when $h(t) = 5$ cm.

RELATING EQN.: $V(t) = \frac{1}{3}\pi r^2(t) \cdot h(t)$ (* not expected to memorize volume of cone)

$$V'(t) = \frac{\pi}{3}\left(2r(t)r'(t)h(t) + r^2(t)\cdot h'(t)\right)$$

Issue here ... We don't know $r'(t)$!

This means we need to subs. $r(t)$.



The two right triangles are similar (both have the same angles) $\Rightarrow \frac{r(t)}{h(t)} = \frac{3}{10} \Rightarrow r(t) = \frac{3h(t)}{10}$

subs. into $V(t)$ formula above

$$V(t) = \frac{\pi}{3}\left(\frac{3h(t)}{10}\right)^2 \cdot h(t)$$

so $V(t) = \frac{3\pi}{100} h^3(t)$

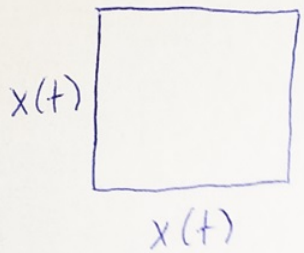
diff. w.r.t. t

$$V'(t) = \frac{9\pi}{100} h^2(t)\, h'(t)$$

subs. $V'(t) = 2$, $h(t) = 5$ $\Rightarrow 2 = \frac{9\pi}{100}\cdot 5^2 h'(t) \Rightarrow 2 = \frac{9\pi}{4} h'(t)$

$$\Rightarrow h'(t) = \frac{8}{9\pi} \approx 0.2829\ cm/sec.$$

Problem 4. Each side of a square is increasing at a rate of 6 cm/s. At what rate is the area of the square increasing when the area of the square is 16 cm^2 ?



$x(t)$ = side length of square at t sec.

$A(t)$ = area of square at t sec.

GIVEN: $x'(t) = 6 \text{ cm/s}$

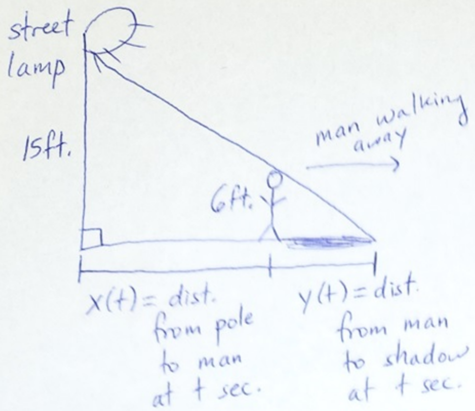
GOAL: Find $A'(t)$ when $A(t) = 16 \text{ cm}^2$.

RELATING EQN: $A(t) = x^2(t)$ (area of square)

$$\begin{aligned} A'(t) &= 2x(t) \cdot x'(t) \\ &= 2(4)(6) \\ &= 48 \text{ cm}^2/\text{s} \end{aligned}$$

when $A = 16$
$\Rightarrow x^2 = 16$
$\Rightarrow x = 4$

Problem 5. A street light is mounted at the top of a 15-feet-tall pole. A man 6 feet tall walks away from the pole with a speed of 5 feet per second along a straight path. How fast is the tip of his shadow moving when he is 40 feet from the pole?



GIVEN: $x'(t) = 5$ ft/sec.

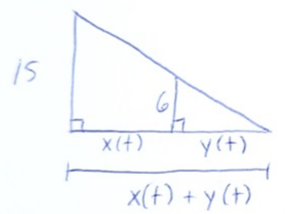
GOAL: Find $\frac{d}{dt}(x(t) + y(t))$ when $x(t) = 40$ ft.

(NOTE: dist. from tip of man's shadow to pole is $x(t) + y(t)$
$\Rightarrow$ Want to find $\frac{d}{dt}(x(t) + y(t))$)

Note: $\frac{d}{dt}(x(t) + y(t)) = x'(t) + y'(t)$
$= 5 + y'(t)$

unknown rate

How can we relate $x(t)$ and $y(t)$?



$$\frac{x+y}{15} = \frac{y}{6} \Rightarrow 6(x+y) = 15y$$
$$\Rightarrow 6x = 9y$$
$$\Rightarrow \boxed{y = \frac{2}{3}x}$$

This is the relating eqn.!

RELATING EQN: $y(t) = \frac{2}{3}x(t)$

$$\Rightarrow y'(t) = \frac{2}{3}x'(t) = \frac{2}{3}(5) = \frac{10}{3} \text{ ft/sec.}$$

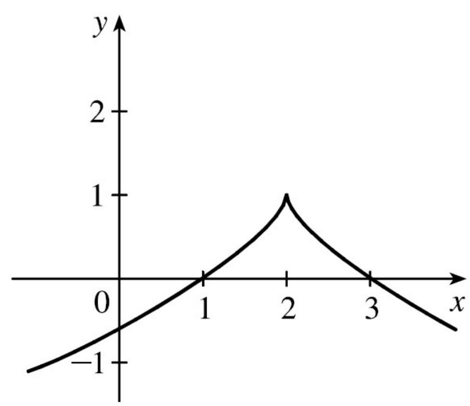
Then $\frac{d}{dt}(x(t) + y(t)) = 5 + \frac{10}{3} = \frac{25}{3} \approx 8.33$ ft/sec.

NOTE: We actually did not need to use that $x(t) = 40$ ft.
Why? It turns out that $x(t) + y(t)$ is constant regardless of the man's distance from the pole.

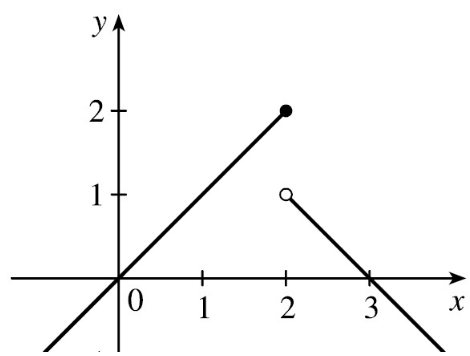
Section 4.1: Maximum & Minimum Values

Problem 6.
(a) Sketch the graph of a function that has a local maximum at 2 and is continuous, but not differentiable at 2.
(b) Sketch the graph of a function that has a local maximum at 2 and is not continuous at 2.
(c) Sketch the graph of a function on $[0, 4]$ that has an absolute maximum, no local maximum, and no absolute minimum.

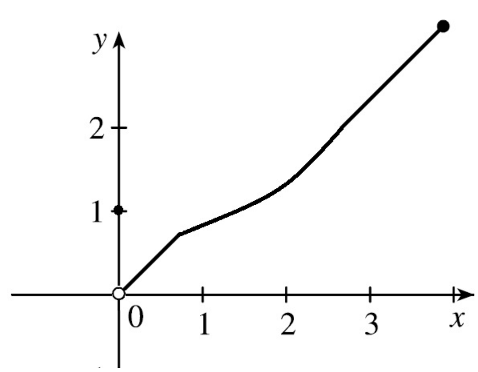
(a)



(b)



(c)



Problem 7. Find the critical numbers of the function.

(a) $f(x) = \dfrac{x^2+2}{2x-1}$ (b) $F(x) = x^{4/5}(x-4)^2$ (c) $g(x) = xe^x$

(a) We have

$$f'(x) = \frac{2x(2x-1)-(x^2+2)(2)}{(2x-1)^2} = \frac{4x^2-2x-2x^2-4}{(2x-1)^2} = \frac{2x^2-2x-4}{(2x-1)^2} = \frac{2(x-2)(x+1)}{(2x-1)^2}$$

Note that $f'(x) = 0$ when $x = -1, 2$ and that $f'(x)$ does not exist when $x = 1/2$. However, since $1/2$ is not in the domain of f, then it cannot be a critical number. Then the only critical numbers of f are -1 and 2.

(b)

$$F(x) = x^{4/5}(x-4)^2 \quad \Rightarrow$$

$$F'(x) = x^{4/5} \cdot 2(x-4) + (x-4)^2 \cdot \tfrac{4}{5}x^{-1/5} = \tfrac{1}{5}x^{-1/5}(x-4)[5 \cdot x \cdot 2 + (x-4) \cdot 4]$$

$$= \frac{(x-4)(14x-16)}{5x^{1/5}} = \frac{2(x-4)(7x-8)}{5x^{1/5}}$$

$F'(x) = 0 \quad \Rightarrow \quad x = 4, \frac{8}{7}$. $F'(0)$ does not exist. Thus, the three critical numbers are 0, $\frac{8}{7}$, and 4.

(c) We have

$$g'(x) = 1 \cdot e^x + xe^x = e^x(1+x).$$

Note that since $e^x > 0$, then $g'(x) = 0$ only when $x = -1$. Since -1 is in $\text{dom}(g)$, it is a critical value.

Problem 8. Find the absolute maximum and the absolute minimum value(s) of the function

$$f(\theta) = 1 + \cos^2(\theta)$$

in the interval $[\pi/4, \pi]$.

We have

$$f'(\theta) = -2\cos(\theta)\sin(\theta).$$

Let us find the critical numbers of f. Since the domains of $\cos(\theta)$ and $\sin(\theta)$ are $(-\infty, \infty)$, then $\text{dom}(f') = (-\infty, \infty)$, meaning that f' is defined everywhere. We solve $-2\cos(\theta)\sin(\theta) = 0$. We have

$$\cos(\theta) = 0 \qquad \text{OR} \qquad \sin(\theta) = 0.$$

In the interval $[\pi/4, \pi]$, $\cos(\theta) = 0$ when $\theta = \pi/2$ and $\sin(\theta) = 0$ when $\theta = \pi$. Then $\pi/2$ and π are the only critical numbers of f. Evaluating f at its critical numbers and its endpoints, we have

$$f(\pi/4) = 1 + \cos^2(\pi/4) = 1 + \left(\frac{\sqrt{2}}{2}\right)^2 = 1 + 2/4 = 3/2,$$

$$f(\pi/2) = 1 + \cos^2(\pi/2) = 1 + (0)^2 = 1 + 0 = 1,$$

$$f(\pi) = 1 + \cos^2(\pi) = 1 + (-1)^2 = 1 + 1 = 2.$$

Therefore, the absolute maximum value is 2 and the absolute minimum value is 1.