MAT 1500 (Dr. Fuentes) — Worksheet 9 - Sections 4.3 & 4.5

Problem 1. Consider the function
$$f(x) = \frac{2x^2}{x^2-1}.$$

(a) Find the domain of f.

$$\text{dom}(f) = \{x \mid x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$$

(b) Find the x and y-intercepts of f.

The x and y intercepts are both 0, meaning that $(0, 0)$ is a point on the graph.

(c) Find the vertical and the horizontal asymptotes of f.

Since $x = \pm 1$ causes f to be of the form $2/0$, then $x = -1$ and $x = 1$ are both vertical asymptotes of f.

To find the horizontal asymptotes, we determine $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. We have

$$\lim_{x\to\infty} \frac{2x^2}{x^2-1} = \lim_{x\to\infty} \frac{2x^2}{x^2} = \lim_{x\to\infty} 2 = 2.$$

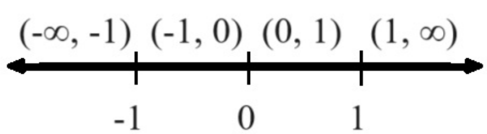
Similarly, $\lim_{x\to-\infty} \frac{2x^2}{x^2-1} = 2$. Then $y = 2$ is the only horizontal asymptote of f.

(d) Find the intervals over which f is increasing and the intervals over which f is decreasing. Use the domain and the critical numbers of f to help you find which intervals to consider.

$f'(x) = \dfrac{-4x}{(x^2-1)^2}$

Since $f'(x) = 0$ when $x = 0$, then 0 is a critical number of f. Note that $x = \pm 1$ are not critical numbers, since they are not in the domain of f, BUT we do need to consider the behavior of f before and after -1 and 1.

Note that the denominator of the derivative is always positive. Therefore, we only need to determine the sign of the numerator.



	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
$\frac{-4x}{+}$	$\frac{-\cdot-}{+} = +$	$\frac{-\cdot-}{+} = +$	$\frac{-\cdot+}{+} = -$	$\frac{-\cdot+}{+} = -$

Therefore, f is increasing on $(-\infty, -1)$ and $(-1, 0)$ and is decreasing on $(0, 1)$ and $(1, \infty)$.

(e) Find the points at which f has a local maximum or a local minimum.

Since 0 is a critical number of f and its derivative changes from positive to negative at 0, then $f(0) = 0$ is a local maximum value of f.

(f) Find the inflection points of f (not just the x-values, but the y-coordinates as well).

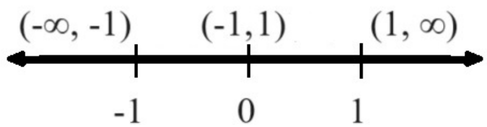
$$f''(x) = \frac{4(3x^2+1)}{(x^2-1)^3}$$

To find the inflection points, we solve $f''(x) = 0$. This gives us $3x^2 + 1 = 0$, or equivalently, $x^2 = -1/3$, which is impossible. This means that f does not have any inflection points.

(g) Find the intervals over which f is concave up and the intervals over which f is concave down.

Since ± 1 are values that are not in the domain of f, we need to determine the concavity of f before and after these values.

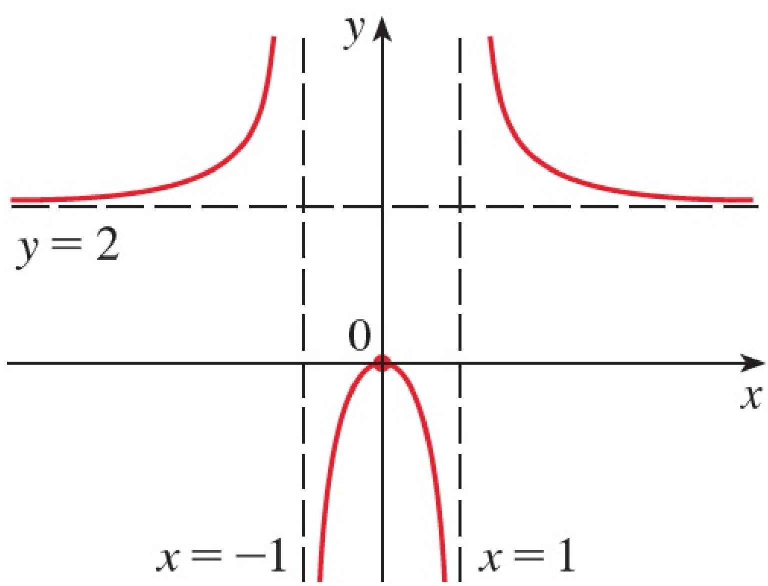
Note that the numerator of the second derivative is always positive.



	$(-\infty,-1)$	$(-1,1)$	$(1,\infty)$
$\frac{+}{(x^2-1)^3}$	$\frac{+}{+} = +$	$\frac{+}{-} = -$	$\frac{+}{+} = -$

Therefore, f is concave up on on $(-\infty,-1)$ and $(1,\infty)$ and is concave down on $(-1,1)$.

(h) Use all of the information from the previous parts to sketch a graph of f. <u>Please label the intercepts, the horizontal asymptote, the local max/min points, and the inflection points on your graph.</u>



Problem 2. Consider the function

$$f(x) = \frac{x^2}{\sqrt{x+1}}.$$

(a) Find the domain of f.

$$\text{dom}(f) = \{x \mid x+1 > 0\} = \{x \mid x > -1\} = (-1, \infty)$$

(b) Find the x and y-intercepts of f.

The x and y intercepts are both 0, meaning that $(0, 0)$ is a point on the graph.

(c) Find the vertical and the horizontal asymptotes of f.

Since $x = -1$ causes f to be of the form $1/0$, then $x = -1$ is the vertical asymptote of f.

To find the horizontal asymptotes, we determine $\lim_{x\to\infty} f(x)$ (note that since x cannot be smaller than -1, we do not need to determine $\lim_{x\to-\infty} f(x)$). We have

$$\lim_{x\to\infty} \frac{x^2}{\sqrt{x+1}} = \lim_{x\to\infty} \frac{x^2}{x^{1/2}} = \infty.$$

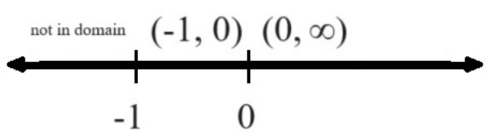
Then f has no horizontal asymptotes.

(d) Find the intervals over which f is increasing and the intervals over which f is decreasing. Use the domain and the critical numbers of f to help you find which intervals to consider.

$$f'(x) = \frac{x(3x+4)}{2(x+1)^{3/2}}$$

Since $f'(x) = 0$ when $x = 0, -4/3$, then 0 is a critical number of f ($-4/3$ is not in the domain of f). Note that $x = -1$ is not a critical number, since it is not in the domain of f.

Note that the denominator of the derivative is always positive, since $x > -1$. Therefore, we only need to determine the sign of the numerator.



$(-1, 0)$	$(0, \infty)$
$\frac{-\cdot+}{+} = -$	$\frac{+\cdot+}{+} = +$

Therefore, f is increasing on $(0, \infty)$ is decreasing on $(-1, 0)$.

(e) Find the points at which f has a local maximum or a local minimum.

Since 0 is a critical number of f and its derivative changes from negative to positive at 0, then $f(0) = 0$ is a local minimum value of f.

(f) Find the inflection points of f (not just the x-values, but the y-coordinates as well).

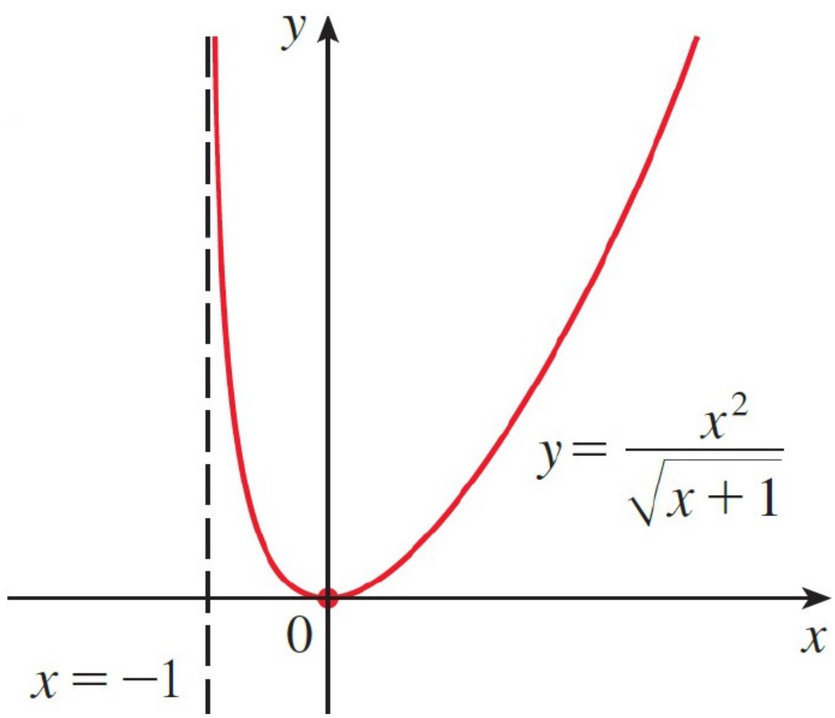
$f''(x) = \dfrac{3x^2 + 8x + 8}{4(x+1)^{5/2}}$

To find the inflection points, we solve $f''(x) = 0$. This gives us $3x^2 + 8x + 8 = 0$, or equivalently, $3x^2 = -8x + 8$, which is impossible, since $-8x + 8 < -8(-1) + 8 = 0$ (remember, the domain of f requires $x > -1$). Therefore, f has no inflection points.

(g) Find the intervals over which f is concave up and the intervals over which f is concave down.

Notice that since $x > -1$, the denominator of the second derivative is always positive. For the same reason, the numerator is also always positive, since $3x^2 + 8x + 8 > 3x^2 + 8(-1) + 8 \geq 0$. Therefore, $f''(x) > 0$ for all $x > -1$, meaning that f is always concave upward (on $(-1, \infty)$).

(h) Use all of the information from the previous parts to sketch a graph of f. Please label the intercepts, the horizontal asymptote, the local max/min points, and the inflection points on your graph.



Problem 3. Consider the function
$$f(x) = \ln(4 - x^2).$$

Domain The domain is

$$\{x \mid 4 - x^2 > 0\} = \{x \mid x^2 < 4\} = \{x \mid |x| < 2\} = (-2, 2)$$

Intercepts The y-intercept is $f(0) = \ln 4$. To find the x-intercept we set

$$y = \ln(4 - x^2) = 0$$

We know that $\ln 1 = 0$, so we have $4 - x^2 = 1 \quad \Rightarrow \quad x^2 = 3$ and therefore the x-intercepts are $\pm\sqrt{3}$.

Asymptotes We look for vertical asymptotes at the endpoints of the domain. Since $4 - x^2 \to 0^+$ as $x \to 2^-$ and also as $x \to -2^+$, we have

$$\lim_{x \to 2^-} \ln(4 - x^2) = -\infty \qquad \lim_{x \to -2^+} \ln(4 - x^2) = -\infty$$

Thus the lines $x = 2$ and $x = -2$ are vertical asymptotes.

Intervals of Increase or Decrease

$$f'(x) = \frac{-2x}{4 - x^2}$$

Since $f'(x) > 0$ when $-2 < x < 0$ and $f'(x) < 0$ when $0 < x < 2$, f is increasing on $(-2, 0)$ and decreasing on $(0, 2)$.

Local Maximum or Minimum Values The only critical number is $x = 0$. Since f' changes from positive to negative at 0, $f(0) = \ln 4$ is a local maximum by the First Derivative Test.

Concavity and Points of Inflection

$$f''(x) = \frac{(4 - x^2)(-2) + 2x(-2x)}{(4 - x^2)^2} = \frac{-8 - 2x^2}{(4 - x^2)^2}$$

Since $f''(x) < 0$ for all x, the curve is concave downward on $(-2, 2)$ and has no inflection point.