MAT 2500 (Dr. Fuentes) Worksheet 8 - (PART 1) - Section 14.2

Section 14.2: Limits & Continuity of 2-Variable Functions

Problem 1. Find the following limits (if they exist).

(a) $\displaystyle\lim_{(x,y)\to(\pi,\pi/2)} \frac{\cos(y)-\sin(2y)}{\cos(x)\cos(y)}$. (b) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{y-x}{1-y+\ln(x)}$.

(c) $\displaystyle\lim_{(x,y)\to(1,1)} (x-1)^2\cos\left(\frac{1}{y}\right)$. **Hint:** Use the Squeeze Theorem.

(a) Let

$$f(x,y)=\frac{\cos(y)-\sin(2y)}{\cos(x)\cos(y)}.$$

Since $f(\pi,\pi/2)$ is undefined, we will try to simplify the function f first. For this limit, we can apply the following trigonometric identity (*you are not expected to memorize trig. identities in this course*):

$$\sin(2y)=2\sin(y)\,\cos(y).$$

We have

$$\begin{aligned}
\lim_{(x,y)\to(\pi,\pi/2)} \frac{\cos(y)-\sin(2y)}{\cos(x)\cos(y)} &= \lim_{(x,y)\to(\pi,\pi/2)} \frac{\cos(y)-2\sin(y)\,\cos(y)}{\cos(x)\cos(y)}\\
&= \lim_{(x,y)\to(\pi,\pi/2)} \frac{\cos(y)-2\sin(y)\,\cos(y)}{\cos(x)\cos(y)}\\
&= \lim_{(x,y)\to(\pi,\pi/2)} \frac{\cos(y)(1-2\sin(y))}{\cos(x)\cos(y)}\\
&= \lim_{(x,y)\to(\pi,\pi/2)} \frac{1-2\sin(y)}{\cos(x)} \quad (\text{since } \cos(y)\neq 0, \text{ as } y\to\pi/2, \text{ but } y\neq\pi/2)\\
&= \frac{1-2\sin(\pi/2)}{\cos(\pi)} = \frac{1-2}{-1} = 1.
\end{aligned}$$

(b) Let

$$g(x,y)=\frac{y-x}{1-y+\ln(x)}.$$

Since $g(1,1)$ is undefined (remember that $\ln(1)=0$) and we cannot further simplify the function g, we will attempt to show that the limit does not exist. Let us find two paths, C_1 and C_2, such that as the point (x,y) travels along these paths, the function g approaches distinct values.

Let C_1 be the path where the point (x,y) approaches the point $(1,1)$ along the line $x=1$. Then since

$$g(1,y)=\frac{y-1}{1-y+\ln(1)}=\frac{y-1}{1-y},$$

we have

$$\lim_{(x,y)\to(1,1)} \frac{y-x}{1-y+\ln(x)} = \lim_{y\to1}\frac{y-1}{1-y} = \lim_{y\to1}\frac{y-1}{-(y-1)} = \lim_{y\to1} -1 = -1,$$

since $y\to1$, but $y\neq1$.

Let C_2 be the path where the point (x, y) approaches the point $(1, 1)$ along the line $x = e$. Then since

$$g(e, y) = \frac{y - e}{1 - y + \ln(e)} = \frac{y - e}{1 - y + 1} = \frac{y - e}{2 - y},$$

we have

$$\lim_{(x,y)\to(1,1)} \frac{y - x}{1 - y + \ln(x)} = \lim_{y\to 1} \frac{y - e}{2 - y} = 1 - e \approx -1.718.$$

Since g has two different limits along two different lines, **the limit DNE** (does not exist).

(c) Let

$$F(x, y) = (x - 1)^2 \cos\left(\frac{1}{y}\right).$$

Note that since

$$-1 \leq \cos(\theta) \leq 1 \quad \Leftrightarrow \quad |\cos(\theta)| \leq 1,$$

for **any** angle θ, we have

$$\left|(x - 1)^2 \cos\left(\frac{1}{y}\right)\right| = \left|(x - 1)^2\right| \left|\cos\left(\frac{1}{y}\right)\right| \leq \left|(x - 1)^2\right| = (x - 1)^2.$$

Then

$$\left|(x - 1)^2 \cos\left(\frac{1}{y}\right)\right| \leq (x - 1)^2 \quad \Leftrightarrow \quad -(x - 1)^2 \leq (x - 1)^2 \cos\left(\frac{1}{y}\right) \leq (x - 1)^2.$$

We have found a way to "squeeze" the function $F(x, y)$ between two functions. Note that

$$\lim_{(x,y)\to(1,1)} (x - 1)^2 = \lim_{x\to 1} (x - 1)^2 = (1 - 1)^2 = 0,$$

and similarly, $\lim_{(x,y)\to(1,1)} -(x - 1)^2 = 0$. Therefore, **by the Squeeze Theorem,**

$$\lim_{(x,y)\to(1,1)} (x - 1)^2 \cos\left(\frac{1}{y}\right) = 0.$$

Problem 2. Determine the set of points at which the function is continuous.

(a) $F(x,y) = \dfrac{xy}{1+e^{x-y}}$.

(b) $g(x,y) = \dfrac{e^x + e^y}{e^{xy} - 1}$

(c) $f(x,y) = \begin{cases} \frac{xy}{x^2+xy+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 \quad \text{if } (x,y) = (0,0). \end{cases}$

(a) Note that $F(x,y) = \frac{xy}{1+e^{x-y}}$ consists of the quotient of a polynomial, $f(x,y) = xy$, and an exponential function, $g(x,y) = 1 + e^{x-y}$, which are both continuous on $\mathbb{R}^2$ (polynomials and exponential functions are continuous everywhere). Since $e^{x-y} > 0$ for **any** values of x and y, then $1 + e^{x-y} \neq 0$ for all values of x and y. Therefore, F has no discontinuities, i.e., F is continuous on $\mathbb{R}^2$.

(b) Note that $g(x,y) = \frac{e^x+e^y}{e^{xy}-1}$ consists of the quotient of two exponential functions, which are both continuous on $\mathbb{R}^2$ (exponential functions are continuous everywhere). Since

$$e^{xy} - 1 = 0 \qquad \Leftrightarrow \qquad e^{xy} = 1 \qquad \Leftrightarrow \qquad xy = \ln(1) = 0,$$

then g has a discontinuities at any point (x,y) that satisfies $xy = 0$, which is only when $x = 0$ or $y = 0$ (this also includes the case when both are equal to 0). Therefore, g is continuous on the set

$$\{(xy) \mid x \neq 0 \text{ OR } y \neq 0\} = \mathbb{R}^2 \setminus \{(x,y) \mid x = 0 \text{ OR } y = 0\}.$$

(c) Let

$$h(x,y) = \frac{xy}{x^2 + xy + y^2}.$$

Note that h is a rational function and hence, is continuous everywhere on its domain. Note that

$$\text{dom}(h) = \left\{(x,y) \mid x^2 + xy + y^2 \neq 0\right\} = \{(0,0)\},$$

that is, h is continuous everywhere except at the point $(0,0)$. Since $f(x,y) = \begin{cases} h(x,y) & \text{if } (x,y) \neq (0,0) \\ 0 \quad \text{if } (x,y) = (0,0), \end{cases}$ then f itself may be continuous everywhere, except possibly at $(0,0)$. We will show that this is indeed the case by showing that $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE.

Let C_1 be the path where the point (x,y) approaches the point $(0,0)$ along the line $y = 0$. Since

$$f(x,0) = \frac{0}{x^2} = 0,$$

we have

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + xy + y^2} = \lim_{x\to 0} 0 = 0.$$

Let C_2 be the path where the point (x, y) approaches the point $(0, 0)$ along the line $y = x$. Since

$$f(x, x) = \frac{x^2}{x^2 + x^2 + x^2} = \frac{x^2}{3x^2} = \frac{1}{3},$$

since $x \to 0$, but $x \neq 0$, we have

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + xy + y^2} = \lim_{x\to 0} \frac{1}{3} = \frac{1}{3}.$$

Since f has two different limits along two different lines, **the limit DNE as $(\mathbf{x}, \mathbf{y}) \to \mathbf{0}$**, which implies that f is **not continuous at** $(,)$.