Section 8.1: Arc Length

Problem 1. Find the exact length of the curve for the given range.

(a) $x = e^y + \frac{1}{4}e^{-y}$, $0 \le y \le 1$, (b) $f(x) = \frac{x^3}{6} + \frac{1}{2x}$, $1 \le x \le 2$, (c) $x^2 - 4y = 2\ln(x)$, $1 \le x \le 2$.

(a)

$$x = e^y + \tfrac{1}{4}e^{-y} \quad \Rightarrow \quad dx/dy = e^y - \tfrac{1}{4}e^{-y} \quad \Rightarrow$$

$$1 + (dx/dy)^2 = 1 + \left(e^y - \tfrac{1}{4}e^{-y}\right)^2 = 1 + \left(e^{2y} - \tfrac{1}{2} + \tfrac{1}{16}e^{-2y}\right) = e^{2y} + \tfrac{1}{2} + \tfrac{1}{16}e^{-2y} = \left(e^y + \tfrac{1}{4}e^{-y}\right)^2. \text{ So}$$

$$L = \int_0^1 \sqrt{\left(e^y + \tfrac{1}{4}e^{-y}\right)^2}\, dy = \int_0^1 \left|e^y + \tfrac{1}{4}e^{-y}\right|\, dy = \int_0^1 \left(e^y + \tfrac{1}{4}e^{-y}\right)\, dy = \left[e^y - \tfrac{1}{4}e^{-y}\right]_0^1$$

$$= e - \frac{1}{4}e^{-1} - \left(1 - \frac{1}{4}\right) = e - \frac{1}{4e} - \frac{3}{4}$$

(b)

$$y = \frac{x^3}{6} + \frac{1}{2x} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{x^2}{2} - \frac{1}{2x^2} \quad \Rightarrow$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}\right) = \frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4} = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2. \text{ So}$$

$$L = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^2 \left|\frac{x^2}{2} + \frac{1}{2x^2}\right|\, dx = \int_1^2 \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)\, dx$$

$$= \left[\frac{1}{6}x^3 - \frac{1}{2x}\right]_1^2 = \left(\frac{8}{6} - \frac{1}{4}\right) - \left(\frac{1}{6} - \frac{1}{2}\right) = \frac{7}{3} + \frac{1}{8} = \frac{17}{12}$$

(c)

$$y = \frac{1}{4}x^2 - \frac{1}{2}\ln x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}x - \frac{1}{2x} \quad \Rightarrow$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{1}{4}x^2 - \frac{1}{2} + \frac{1}{4x^2}\right) = \frac{1}{4}x^2 + \frac{1}{2} + \frac{1}{4x^2} = \left(\frac{1}{2}x + \frac{1}{2x}\right)^2. \text{ So}$$

$$L = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^2 \left|\frac{1}{2}x + \frac{1}{2x}\right|\, dx = \int_1^2 \left(\frac{1}{2}x + \frac{1}{2x}\right)\, dx$$

$$= \left[\frac{1}{4}x^2 + \frac{1}{2}\ln|x|\right]_1^2 = \left(1 + \frac{1}{2}\ln 2\right) - \left(\frac{1}{4} + 0\right) = \frac{3}{4} + \frac{1}{2}\ln 2$$

Problem 2. Find the arc length function for the curve $y = \arcsin(x) + \sqrt{1-x^2}$ with starting point $(0, 1)$.

$$y = \sin^{-1} x + \sqrt{1-x^2} \quad \Rightarrow \quad y' = \frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} = \frac{1-x}{\sqrt{1-x^2}} \quad \Rightarrow$$

$$1 + (y')^2 = 1 + \frac{(1-x)^2}{1-x^2} = \frac{1-x^2+1-2x+x^2}{1-x^2} = \frac{2-2x}{1-x^2} = \frac{2(1-x)}{(1+x)(1-x)} = \frac{2}{1+x} \quad \Rightarrow$$

$\sqrt{1+(y')^2} = \sqrt{\frac{2}{1+x}}$. Thus, the arc length function with starting point $(0, 1)$ is given by

$$s(x) = \int_0^x \sqrt{1+[f'(t)]^2}\,dt = \int_0^x \sqrt{\frac{2}{1+t}}\,dt = \sqrt{2}\left[2\sqrt{1+t}\right]_0^x = 2\sqrt{2}\left(\sqrt{1+x} - 1\right).$$

Problem 3. A hawk flying at $15\,m/s$ at an altitude of $180\,m$ accidentally drops its prey. The parabolic trajectory of the falling prey is described by the equation

$$y = 180 - \frac{x^2}{45}$$

until it hits the ground, where **y is its height above the ground and x is the horizontal distance traveled in meters.**

Calculate the distance traveled by the prey from the time it is dropped until the time it hits the ground. Express your answer correct to the nearest tenth of a meter.

**For Problem 3 you will need the following integration formula, which would be provided on quizzes/exams:

$$\int \sec^3(u)\,\mathrm{du} = \frac{1}{2}\sec(u)\tan(u) + \frac{1}{2}\ln(|\sec(u) + \tan(u)|) + C.$$

The prey hits the ground when $y = 0 \quad \Leftrightarrow \quad 180 - \frac{1}{45}x^2 = 0 \quad \Leftrightarrow \quad x^2 = 45 \cdot 180 \quad \Rightarrow \quad x = \sqrt{8100} = 90$,

since x must be positive. $y' = -\frac{2}{45}x \quad \Rightarrow \quad 1 + (y')^2 = 1 + \frac{4}{45^2}x^2$, so the distance traveled by the prey is

$$L = \int_0^{90} \sqrt{1 + \frac{4}{45^2}x^2}\,dx = \int_0^4 \sqrt{1+u^2}\left(\tfrac{45}{2}\,du\right) \qquad \begin{bmatrix} u = \frac{2}{45}x, \\ du = \frac{2}{45}\,dx \end{bmatrix}$$

$$\overset{21}{=} \tfrac{45}{2}\left[\tfrac{1}{2}u\sqrt{1+u^2} + \tfrac{1}{2}\ln\left(u + \sqrt{1+u^2}\right)\right]_0^4 = \tfrac{45}{2}\left[2\sqrt{17} + \tfrac{1}{2}\ln\left(4+\sqrt{17}\right)\right] = 45\sqrt{17} + \tfrac{45}{4}\ln\left(4+\sqrt{17}\right) \approx 209.1 \text{ m}$$