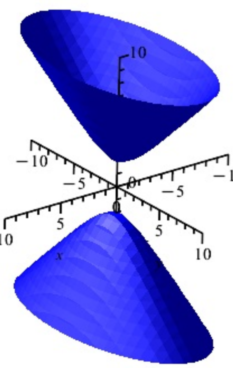
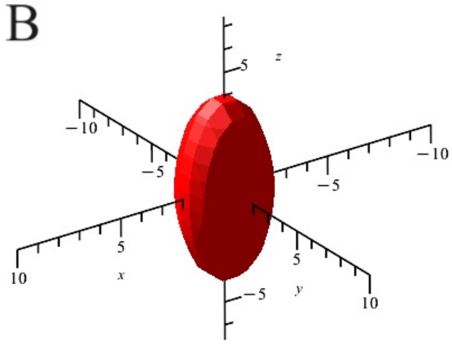
Problem 1. Use traces to sketch and identify the surface.

(a) $\dfrac{x^2}{2^2} + \dfrac{y^2}{2^2} + \dfrac{z^2}{4^2} = 1$ (b) $z^2 - 4x^2 - y^2 = 4$ (c) $x = y^2 - z^2$

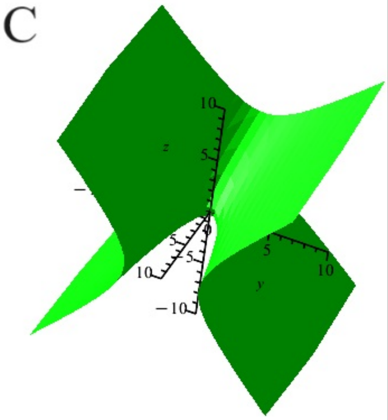
A



B



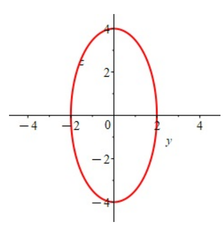
C



(a) The trace of the surface with the plane $x = 0$ is

$$\frac{y^2}{2^2} + \frac{z^2}{4^2} = 1,$$

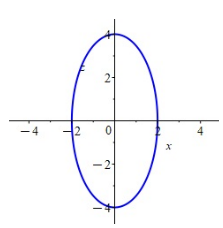
which is an ellipse with horizontal radius 2 and vertical radius 4 on the yz-plane, as shown below.



The trace of the surface with the plane $y = 0$ is

$$\frac{x^2}{2^2} + \frac{z^2}{4^2} = 1,$$

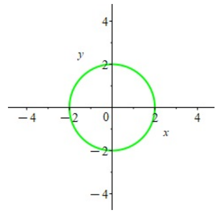
which is an ellipse with horizontal radius 2 and vertical radius 4 on the xz-plane, as shown below.



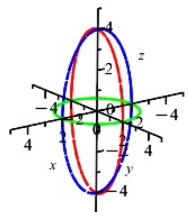
The trace of the surface with the plane $z = 0$ is

$$\frac{x^2}{2^2} + \frac{y^2}{2^2} = 1,$$

which is a circle of radius 2 on the xy-plane, as shown below.



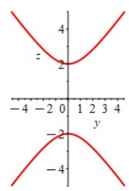
Combining all three trace curves, we obtain the "skeleton" of the surface, shown below. The surface must be B.



(b) The trace of the surface with the plane $x = 0$ is

$$z^2 - y^2 = 4 \Leftrightarrow \frac{z^2}{2^2} - \frac{y^2}{2^2} = 1,$$

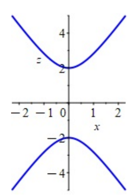
which is the hyperbola on the yz-plane shown below.



The trace of the surface with the plane $y = 0$ is

$$z^2 - 4x^2 = 4 \Leftrightarrow \frac{z^2}{2^2} - \frac{x^2}{1^2} = 1,$$

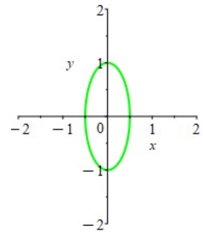
which is the hyperbola on the xz-plane shown below.



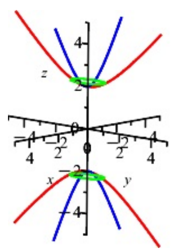
The trace of the surface with the planes $z = \pm\sqrt{5}$ is

$$5 - 4x^2 - y^2 = 4 \Leftrightarrow \frac{x^2}{(1/2)^2} + \frac{y^2}{1^2} = 1,$$

which is an ellipse with horizontal radius $1/2$ and vertical radius 1 on the the plane $z = \pm\sqrt{5}$, as shown below.



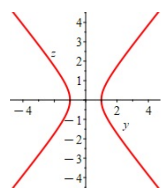
Combining all three trace curves, we obtain the "skeleton" of the surface, shown below. The surface must be A.



(c) The trace of the surface with the plane $x = 1$ is

$$1 = y^2 - z^2$$

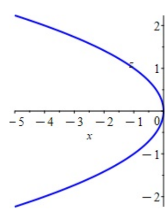
which is a hyperbola on the plane $x = 1$ as shown below.



The trace of the surface with the plane $y = 0$ is

$$x = -z^2$$

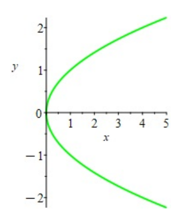
which is a sideways parabola on the xz-plane as shown below.



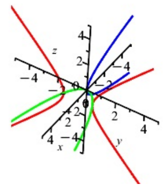
The trace of the surface with the plane $z = 0$ is

$$x = y^2$$

which is a sideways parabola on the xy-plane as shown below.



Combining all three trace curves, we obtain the "skeleton" of the surface, shown below. The surface must be C.



Section 13.1: Vector Functions & Space Curves

Problem 2. Find the domain of the vector function

$$\mathbf{r}(t) = \left\langle \ln(t+1), \frac{t}{\sqrt{9-t^2}}, 2^t \right\rangle.$$

The component functions of $\mathbf{r}$ are

$$f(t) = \ln(t+1), \quad g(t) = \frac{t}{\sqrt{9-t^2}}, \quad \text{and} \quad h(t) = 2^t.$$

We will find the domains of each of f, g, and h.

For the component function f, we have

$$\text{dom}(f) = \{t \mid t+1 > 0\} = \{t \mid t > -1\} = (-1, \infty).$$

For the component function g, we have

$$\text{dom}(g) = \left\{t \mid 9-t^2 > 0\right\} = \left\{t \mid 9 > t^2\right\} = \{t \mid 3 > |t|\} = \{t \mid -3 < t < 3\} = (-3, 3).$$

Since $h(t) = 2^t$ is an exponential function, its domain is

$$\text{dom}(h) = \mathbb{R} = (-\infty, \infty).$$

Since $\text{dom}(\mathbf{r})$ is the intersection of $\text{dom}(f)$, $\text{dom}(g)$, and $\text{dom}(h)$, we have

$$\text{dom}(\mathbf{r}) = (-1, 3).$$

Problem 3. Let

$$\mathbf{r}(t) = te^{-t}\mathbf{i} + \frac{t^3+t}{2t^3-1}\mathbf{j} + \frac{t}{|t|}\mathbf{k}.$$

(a) Find $\lim\limits_{t\to\infty} \mathbf{r}(t)$. (b) Determine if $\mathbf{r}(t)$ continuous at $t = 0$.

(a) The component functions of $\mathbf{r}$ are

$$f(t) = te^{-t}, \quad g(t) = \frac{t^3+t}{2t^3-1}, \quad \text{and} \quad h(t) = \frac{t}{|t|}.$$

We will take the limit as $t \to \infty$ of each of f, g, and h.

Since $te^{-t} = \dfrac{t}{e^t}$ and $e^t \to \infty$ as $t \to \infty$, we have $\frac{\infty}{\infty}$ form. Then by L'Hopital's Rule we have

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} \frac{t}{e^t} = \lim_{t\to\infty} \frac{1}{e^t} = 0.$$

Since $t^3 + t$ and $2t^3 - 1$ are both polynomial expressions of degree 3, we have

$$\lim_{t\to\infty} g(t) = \lim_{t\to\infty} \frac{t^3+t}{2t^3-1} = \lim_{t\to\infty} \frac{t^3}{2t^3} = \frac{1}{2}.$$

Note that another to obtain $\lim_{t\to\infty} g(t)$ is to apply L'Hopital's Rule repeatedly.

Finally, since $t \to \infty$, then $t > 0$, so $|t| = t$. Then we have

$$\lim_{t\to\infty} h(t) = \lim_{t\to\infty} \frac{t}{|t|} = \lim_{t\to\infty} \frac{t}{t} = 1.$$

Therefore, $\lim\limits_{t\to\infty} \mathbf{r}(t) = \langle 0, 1/2, 1 \rangle$.

(b) Since $h(0) = 0/0$ is undefined, the function is not continuous at 0.

Problem 4. Consider the following vector functions.

(i) $\mathbf{r}(t) = \langle 3, t, 2 - t^2 \rangle$ (ii) $\mathbf{r}(t) = 2t\mathbf{i} + 2\cos(t)\mathbf{j} + 3\sin(t)\mathbf{k}$

For each one
(a) determine its space curve **by hand** and use an arrow to indicate the direction in which t increases.
(b) verify that your sketch is correct using the *spacecurve( )* command **in Maple.**

(i) The parametric equations of $\mathbf{r}(t) = \langle 3, t, 2 - t^2 \rangle$ are

$$x = 3, \quad y = t, \quad \text{and} \quad z = 2 - t^2.$$

Since $x = 3$, the space curve of $\mathbf{r}$ lies on the vertical plane $x = 3$. Note that

$$z = 2 - y^2 = -y^2 + 2.$$

Then the space curve consists of a downward parabola shifted up 2 units that lies on the $x = 3$ plane.

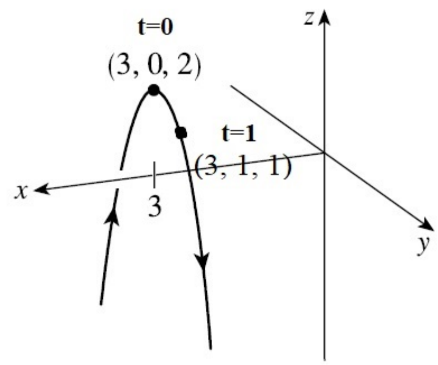
It suffices to find two points on the curve to determine the direction of the space curve. When $t = 0$ we have the point $(3, 0, 2)$, and when $t = 1$ we have the point $(3, 1, 1)$.
The Maple command to obtain a plot of the space curve is shown below.
(ii) The parametric equations of $\mathbf{r}(t) = 2t\mathbf{i} + 2\cos(t)\mathbf{j} + 3\sin(t)\mathbf{k}$ are

$$x = 2t, \quad y = 2\cos(t), \quad \text{and} \quad z = 3\sin(t).$$

Note that

$$\left(\frac{y}{2}\right)^2 + \left(\frac{z}{3}\right)^2 = \cos^2(t) + \sin^2(t) = 1.$$

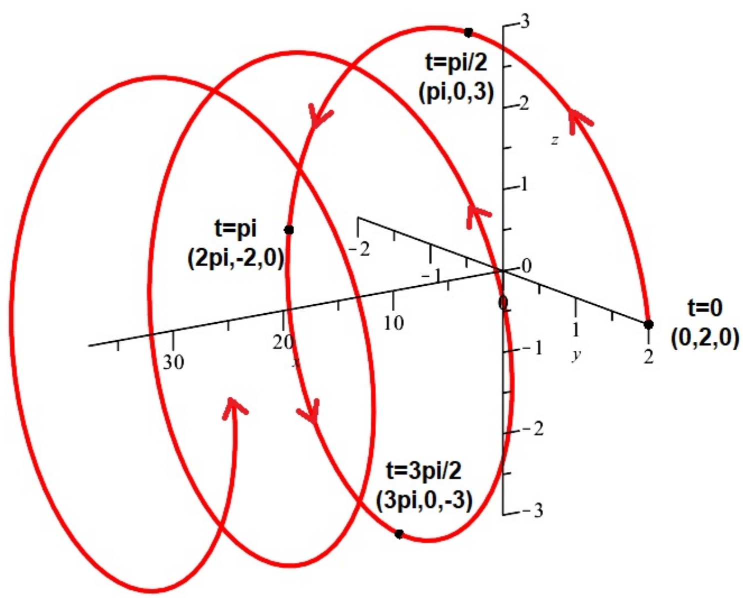
Then $y^2/2^2 + z^2/3^2 = 1$, which is the equation of an elliptic cylinder with horizontal radius 2, vertical radius 3, and with axis the x-axis. This means that the space curve of $\mathbf{r}$ lies on the surface of this elliptic




```
> with(plots) :
> spacecurve([3, t, 2 − t^2], t = −5 ..5, axes = normal, labels = [x, y, z], thickness = 5)
```

cylinder.

By plotting some points for several increasing values of t, we can determine the space curve and its direction. When $t = 0$, we have the point $(0, 2, 0)$, when $t = \pi/2$, we have the point $(\pi.0, 3)$, when $t = \pi$, we have the point $(2\pi, -2, 0)$, and when $t = 3\pi/2$, we have the point $(3\pi, 0, -3)$. Plotting these points along the surface of the cylinder, we see that the space curve is the helix shown below.



The Maple command to obtain a plot of the space curve is shown below.

```
> with(plots) :
> spacecurve([2 t, 2·cos(t), 3·sin(t)], t = 0 ..20, axes = normal, labels = [x, y, z], thickness = 5, color = red)
```

~~**Problem 5.** Let $\mathbf{r}(t) = (t+1)\mathbf{i} + t\mathbf{j} + (2\cos(t+1))\mathbf{k}$.~~
~~(a) Determine and draw the projections of the space curve of **r** onto the three coordinate planes **by hand**.~~
~~(b) Draw a rough sketch of the space curve of **r by hand**.~~
~~(c) Verify your answers in (a) and (b) using the *plot( )* and *spacecurve( )* commands **in Maple.**~~

PROBLEM 5 HAS BEEN OMITTED.

Problem 6.
(a) Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.
HINT: $\cos^2(x) + \sin^2(x) = 1$.

(b) Plot the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$ **in Maple** on the same plot using the *implicitplot3d( )* and the *display( )* commands. Give each surface a distinct color.

(c) Use **Maple** to plot the intersection of the cylinder and the plane using the *intersectplot( )* command.

(d) Use **Maple** to verify that the space curve of the vector function you defined in (a) matches your plot in (c). **Be sure to set the ranges for x, y, and z the same as for the plot in (d).**

First of all, note that the intersection of the circular cylinder and the diagonal plane $y + z = 2$ would be an ellipse (think of the cross-section you would obtain if you sliced the cylinder on a diagonal). We must determine the parametric equations of the vector function **r** that satisy both of the equations $x^2 + y^2 = 1$ and $y + z = 2$ in order to obtain a space curve that is precisely the curve of intersection of the circular cylinder and the plane. If we choose $x = \cos(t)$ and $y = \sin(t)$, then

$$x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1,$$

and so the equation of the cylinder is satisfied. The y and z parametric equations must satisfy $y + z = 2$, or equivalently, $z = 2 - y$. Since $y = \sin(t)$, then we must have

$$z = 2 - \sin(t).$$

Note that in order to avoid infinitely many rotations around the ellipse (the curve of intersection), we need to restrict t to obtain one rotation. We can choose, for example, $0 \le t \le 2\pi$. Then a vector function that represents the curve of intersection is

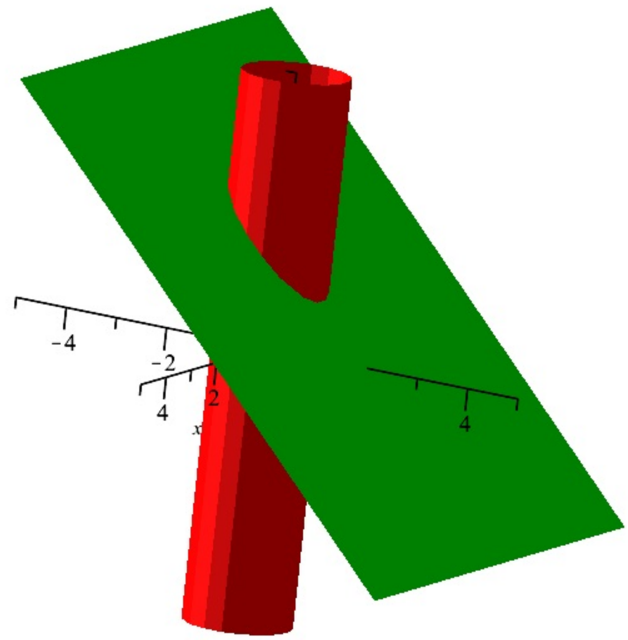
$$\mathbf{r}(t) = (\cos(t))\mathbf{i} + (\sin(t))\mathbf{j} + (2 - \sin(t))\mathbf{k}, \quad 0 \le t \le 2\pi.$$

> $with(plots):$

> $plot1 := implicitplot3d(x^2 + y^2 = 1, x = -5 ..5, y = -5 ..5, z = -5 ..5, axes = normal, labels = [x, y, z], style = surface, color = red):$

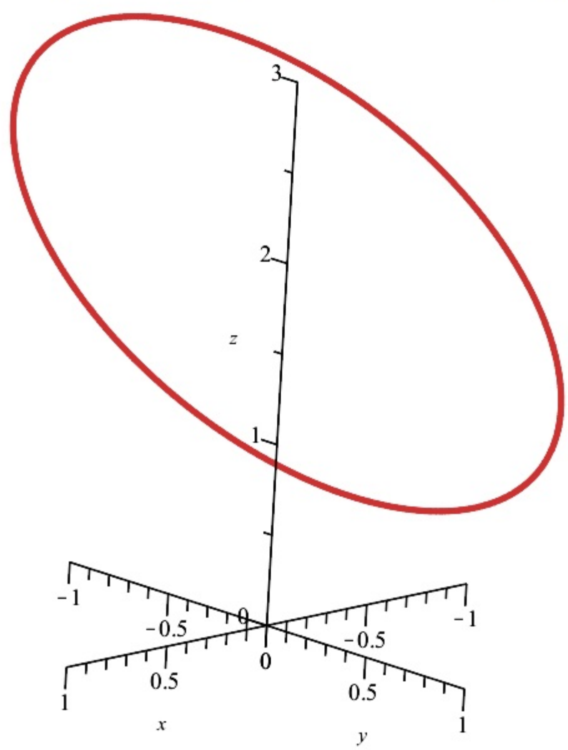
> $plot2 := implicitplot3d(y + z = 2, x = -5 ..5, y = -5 ..5, z = -5 ..5, axes = normal, labels = [x, y, z], style = surface, color = green):$

> $display(plot1, plot2)$

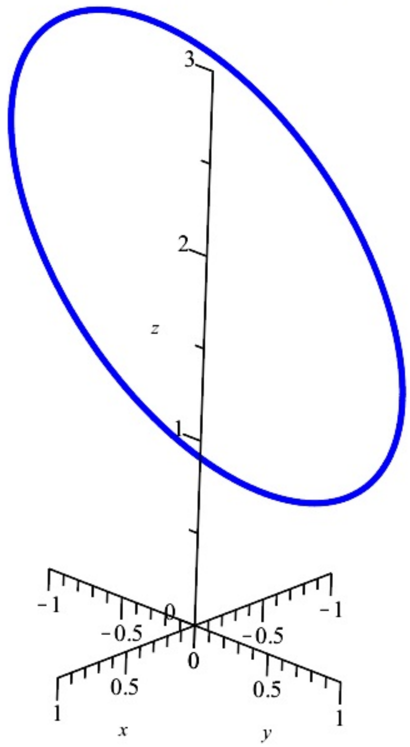


> $with(plots):$

> $intersectplot(x^2 + y^2 = 1, y + z = 2, x = -1 ..1, y = -1 ..1, z = 0 ..3, axes = normal, labels = [x, y, z], style = surface, color = orange, thickness = 5)$



```
> with(plots) :
> spacecurve([cos(t), sin(t), 2 − sin(t)], t = −0 ..2·Pi, axes = normal, labels = [x, y, z], thickness = 5, color = blue)
```

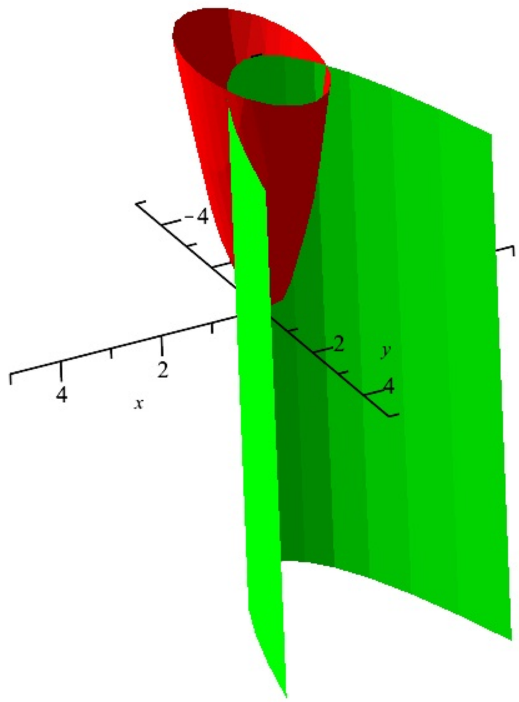


Problem 7. Repeat parts (a)-(d) in Problem 5 for the intersection of the paraboloid $z = 4x^2 + y^2$ and the parabolic cilinder $y = x^2$.
HINT: As the first parametric equation of the curve of intersection, let $x = t$.

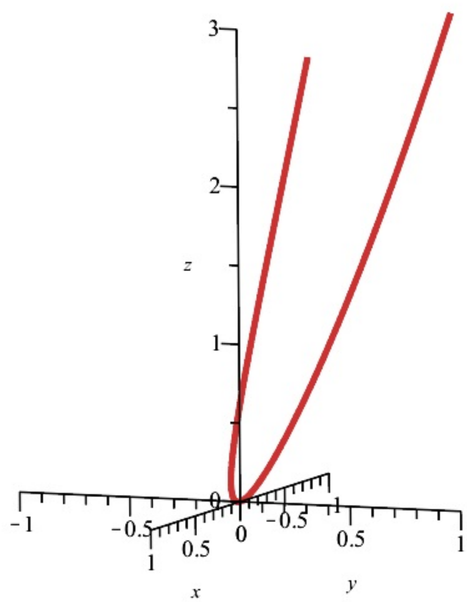
We must determine the parametric equations of the vector function $\mathbf{r}$ that satisy both of the equations $z = 4x^2 + y^2$ and $y = x^2$ in order to obtain a space curve that is precisely the curve of intersection of the paraboloid and the parabolic cilinder. If we choose $x = t$, then $y = t^2$, and then $z = 4t^2 + (t^2)^2 = 4t^2 + t^4$. Then vector function for the curve of intersection is

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + (4t^2 + t^4)\mathbf{k}.$$

```
> with(plots) :
> plot3 := implicitplot3d(z = 4·x^2 + y^2, x = −5 ..5, y = −5 ..5, z = −5 ..5, axes = normal, labels = [x, y, z], style = surface, color = red) :
> plot4 := implicitplot3d(y = x^2, x = −5 ..5, y = −5 ..5, z = −5 ..5, axes = normal, labels = [x, y, z], style = surface, color = green) :
> display(plot3, plot4)
```




```
> intersectplot(z = 4·x^2 + y^2, y = x^2, x = −1 ..1, y = −1 ..1, z = 0 ..3, axes = normal, labels = [x, y, z], style = surface, color = orange, thickness = 5)
```



> $spacecurve\left(\left[t, t^2, 4 \cdot t^2 + t^4\right], t = -1 \,..1, axes = normal, labels = [x, y, z], thickness = 5, color = blue\right)$