Section 2.5: Continuity

Problem 1. Show that there is a solution of the equation $\ln(x) = 3 - 2x$ between 1 and 2.

Let $f(x) = \ln(x) - 3 + 2x$. Showing that f has a zero in the interval $[1,2]$ is equivalent to showing that $\ln(x) = 3 - 2x$ has a solution between 1 and 2.

We want to be able to apply the IVT, so we need to show that f is continuous on $[1,2]$.

↳ Note that $f(x) = g(x) + h(x)$, where $g(x) = \ln(x)$ and $h(x) = -3 + 2x$, both of which are cont. on their domains. Since

$$\mathrm{dom}(g) = (0, \infty) \quad \text{and} \quad \mathrm{dom}(h) = (-\infty, \infty),$$

then f is cont. where BOTH g and h are cont., which is on $(0,\infty)$.

Then f is cont. on $[1,2]$.

Since $f(1) = \ln(1) - 3 + 2\cdot 1 = 0 - 3 + 2 = -1 < 0$ and $f(2) = \ln(2) - 3 + 2\cdot 2 \geq 1.69 > 0$ then $f(1) < 0 < f(2)$, so by the IVT there must be a value c such that $1 < c < 2$ and $f(c) = 0$.

Section 2.6: Limits at Infinity; Horizontal Asymptotes

Problem 2. Find the horizontal asymptotes of the following functions:

(a) $f(x) = \frac{9x^3+8x-4}{(x-2)^3}$ (b) $g(x) = \frac{\sqrt{1+4x^6}}{2-x^3}$.

(a) The leading term of $9x^3+8x-4$ is $9x^3$ and the leading term of $(x-2)^3$ is x^3. Then

$$\lim_{\substack{x\to\infty \\ \text{OR} \\ x\to-\infty}} \frac{9x^3+8x-4}{(x-2)^3} = \lim_{\substack{x\to\infty \\ \text{OR} \\ x\to-\infty}} \frac{9x^3}{x^3} = \lim_{\substack{x\to\infty \\ -\infty}} 9 = 9.$$

The only h.a. of f is $y = 9$.

(b) We need to check $\lim_{x\to\infty} g(x)$ AND $\lim_{x\to-\infty} g(x)$.

$$\lim_{x\to\infty} \frac{\sqrt{1+4x^6}}{2-x^3} = \lim_{x\to\infty} \frac{\sqrt{1+4x^6}}{2-x^3} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\frac{\sqrt{1+4x^6}}{x^3}}{\frac{2}{x^3}-1} = \lim_{x\to\infty} \frac{\frac{\sqrt{1+4x^6}}{\sqrt{x^6}}}{\frac{2}{x^3}-1} = \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^6}+4}}{\frac{2}{x^3}-1} = \rightarrow$$

$= \frac{\sqrt{0+4}}{0-1} = \frac{\sqrt{4}}{-1} = -2.$

$\lim_{x\to-\infty} \frac{\sqrt{1+4x^6}}{2-x^3} =$ SAME STEPS EXCEPT $= \lim_{x\to-\infty} \frac{\frac{\sqrt{1+4x^6}}{-\sqrt{x^6}}}{\frac{2}{x^3}-1} = \lim_{x\to-\infty} \frac{-\sqrt{\frac{1}{x^6}+4}}{\frac{2}{x^3}-1}$

why?

$\sqrt{x^6} = \sqrt{(x^3)^2} = |x^3|$

so if $x\to\infty$, then $x<0$,

so $x^3 = -\sqrt{x^6} = -|x^3|$

$= \frac{-\sqrt{0+4}}{-1} = 2.$

The h.a. of g are $y=-2$ and $y=2$.

Section 2.7: Derivatives and Rates of Change

Problem 3. The height (in meters) of a rock t seconds after it has been thrown upward on the planet Mars is given by

$$H(t) = 10t - 1.86t^2.$$

(a) Find the velocity of the rock after one second.
(b) Find the velocity of the rock when $t = a$.
(c) When will the rock hit the surface?
(d) With what velocity will the rock hit the surface?

(a) Let $v(t)$ be the velocity of the rock after t seconds. Then

vel. after 1 sec. $= v(1) = \lim_{h\to0} \frac{H(1+h)-H(1)}{h} = \lim_{h\to0} \frac{10(h+1)-1.86(h+1)^2-8.14}{h}$

$H(1) = 10-1.86\cdot1^2 = 8.14$

$= \lim_{h\to0} \frac{10h+\cancel{10}-1.86h^2-3.72h-\cancel{1.86}-\cancel{8.14}}{h}$

$= \lim_{h\to0} \frac{10\cancel{h}-1.86h^{\cancel{2}}-3.72\cancel{h}}{\cancel{h}}$

$= \lim_{h\to0} 10-1.86h-3.72 = 10-1.86\cdot0-3.72$

$= \boxed{6.28 \text{ m/sec.}}$

(b) vel. of rock when $t=a$ sec. $= v(a) = \lim_{h\to0} \frac{H(a+h)-H(a)}{h}$

$= \lim_{h\to0} \frac{10(a+h)-1.86(a+h)^2-(10a-1.86a^2)}{h}$

$= \lim_{h\to0} \frac{\cancel{10a}+10h-\cancel{1.86a^2}-3.72ah-1.86h^2-\cancel{10a}-\cancel{1.86a^2}}{h}$

$= \lim_{h\to0} \frac{10\cancel{h}-3.72a\cancel{h}-1.86h^{\cancel{2}}}{\cancel{h}} = \lim_{h\to0} 10-3.72a-1.86h = 10-3.72a-1.86\cdot0$

$= \boxed{10-3.72a \text{ m/s.}}$

$10-3.72a$

(c) The rock will hit the surface when its height from the ground is 0. That is, when $H(t)=0$. Solve $10t-1.86t^2=0 \Rightarrow t(10-1.86t)=0 \Rightarrow t=0$ sec, $t=5.38$ sec.

At $t=0$ is the initial time, so rock hits ground when $t=5.38$ sec.

(d) Rock will hit ground with velocity $v(5.38) = 10.01$ m/s