MAT 2500 (Dr. Fuentes) Worksheet 1 - Sections 12.1 & 12.2

Section 12.1: Three-Dimensional Coordinate Systems

Problem 1. Which of the points $A = (-4, 0, -1)$, $B = (3, 1, -5)$, and $C = (2, 4, 6)$ is closest to the yz-plane? Which points lie in the xz-plane?

Solution: Recall that for any point $P = (a, b, c)$ in $\mathbb{R}^3$,

$$|a| = \text{distance from } P \text{ to the } yz\text{-plane},$$

$$|b| = \text{distance from } P \text{ to the } xz\text{-plane},$$

$$|c| = \text{distance from } P \text{ to the } xy\text{-plane}.$$

Since the point C has the x-coordinate with the smallest absolute value among the three points, C is the closest the the yz-plane.

Since the y-coordinate of A is 0, this means that the distance from A to the xz-plane is 0, meaning that A lies on the xz-plane.

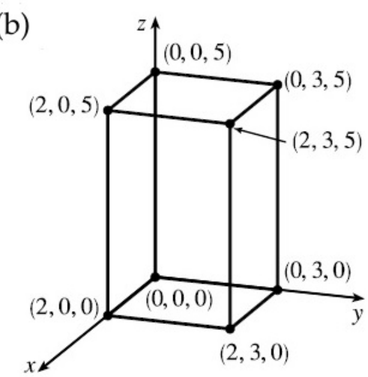
Problem 2.
(a) What are the projections of the point $(2, 3, 5)$ on the $xy-$, $yz-$, and $xz-$planes?
(b) Draw a rectangular box with the origin and $(2, 3, 5)$ as opposite points and with its faces parallel to the coordinate planes. Label all the points of the box.
(c) Find the length of the diagonal of the box.

(a) The projection of $(2, 3, 5)$ onto the xy-plane is $(2, 3, 0)$; onto the yz-plane, $(0, 3, 5)$; onto the xz-plane, $(2, 0, 5)$.

(c) The length of the diagonal of the box is the distance between the origin and $(2, 3, 5)$, given by

$$\sqrt{(2-0)^2 + (3-0)^2 + (5-0)^2} = \sqrt{38} \approx 6.16$$

(b)



Problem 3. Describe and/or sketch the regions of $\mathbb{R}^3$ represented by the following equation(s) or inequalities.

(a) $x^2 + z^2 \le 25$, $0 \le y \le 2$ (b) $1 \le x^2 + 2x + y^2 + z^2 \le 5$ (c) $0 \le x \le 3$, $0 \le y \le 3$, $0 \le z \le 3$.

(a) When $y = 0$, the inequality $x^2 + z^2 \le 25$ represents the set of points on the circumfrence of or within a circle of radius 5 on the xz-plane centered at the origin; that is, we have a solid circle of radius 5 on the xz-plane centered at the origin. However, since $0 \le y \le 2$, the region is solid cylinder of radius 5 and height 2 with the y-axis as its axis.

(b) The inequality $1 \le x^2 + 2x + y^2 + z^2 \le 5$ is equivalent to $1 \le \sqrt{x^2 + 2x + y^2 + z^2} \le \sqrt{5}$, so the region consists of all points (x, y, z) whose distance from the origin $(0, 0, 0)$ is at least 1 and at most $\sqrt{5}$; that is, the region consists of the set of all points between and on the surfaces of a sphere of radius 1 and a sphere of radius 5, both centered at the origin.

(c) Suppose that $z = 0$. Then set of all points $(x, y, 0)$ such that $0 \leq x \leq 3$ and $0 \leq y \leq 3$ forms a region that consists of a solid square on the xy-plane whose sides are of length 3 and corner points are $(0, 0, 0)$, $(3, 0, 0)$, $(3, 3, 0)$, and $(0, 3, 0)$. However, since $0 \leq z \leq 3$, then the region is a cube whose side are of length 3 and the corners of its base are the four previously mentioned points.

Section 12.2: Vectors

Problem 4. Find $\mathbf{a} + \mathbf{b} - \mathbf{c}$, $\mathbf{a} + 2\mathbf{b}$, $|\mathbf{a}|$, and $|\mathbf{a} - \mathbf{c}|$ for the following sets of vectors in $\mathbb{R}^2$:

(a) $\mathbf{a} = \langle 1, 2 \rangle, \mathbf{b} = \langle -2, 3 \rangle, \mathbf{c} = \langle 3, 3 \rangle$ $\qquad$ (b) $\mathbf{a} = 3\mathbf{i} - 5\mathbf{j}, \mathbf{b} = -\mathbf{i} - 2\mathbf{j}, \mathbf{c} = \mathbf{i} + \mathbf{j}$

Illustrate all of the vectors in part (b) geometrically.

(a) We have

$$\mathbf{a} + \mathbf{b} - \mathbf{c} = \langle 1 - 2 - 3, 2 + 3 - 3 \rangle = \langle -4, 2 \rangle,$$
$$\mathbf{a} + 2\mathbf{b} = \langle 1, 2 \rangle + 2\langle -2, 3 \rangle = \langle 1, 2 \rangle + \langle -4, 6 \rangle = \langle 1 - 4, 2 + 6 \rangle = \langle -3, 8 \rangle,$$
$$|\mathbf{a}| = \sqrt{1^2 + 2^2} = \sqrt{5},$$

and

$$|\mathbf{a} - \mathbf{c}| = |\langle 1 - 3, 2 - 3 \rangle| = |\langle -2, -1 \rangle| = \sqrt{(-2)^2 + (-1)^2} = \sqrt{5}.$$

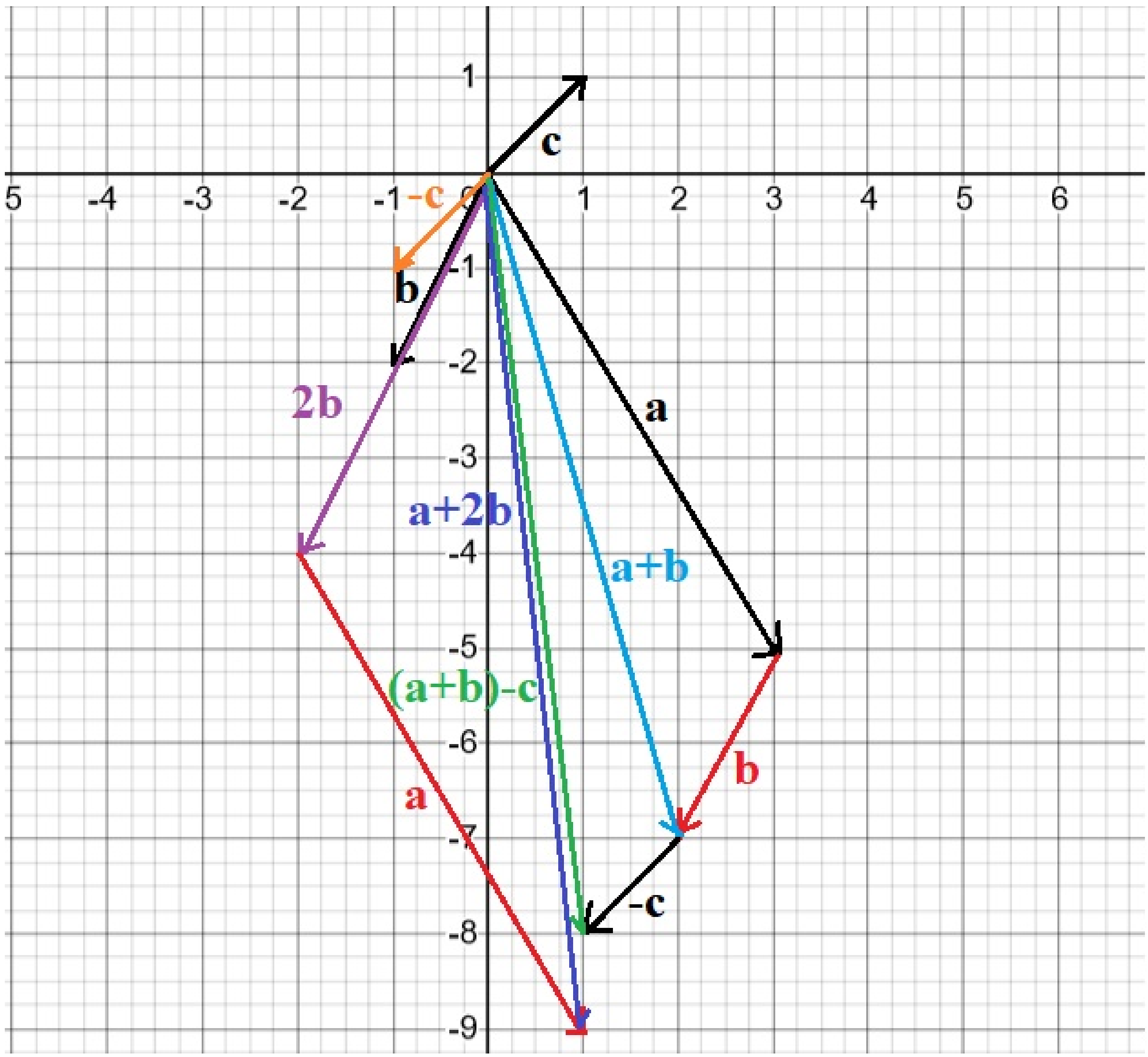
(b) We have

$$\mathbf{a} + \mathbf{b} - \mathbf{c} = (3 + (-1) - 1)\mathbf{i} + (-5 + (-2) - 1)\mathbf{j} = \mathbf{i} - 8\mathbf{j},$$
$$\mathbf{a} + 2\mathbf{b} = (3\mathbf{i} - 5\mathbf{j}) + 2(-\mathbf{i} - 2\mathbf{j}) = 3\mathbf{i} - 5\mathbf{j} - 2\mathbf{i} - 4\mathbf{j} = \mathbf{i} - 9\mathbf{j},$$
$$|\mathbf{a}| = \sqrt{3^2 + (-5)^2} = \sqrt{34},$$

and

$$|\mathbf{a} - \mathbf{c}| = |3\mathbf{i} - 5\mathbf{j} - (\mathbf{i} + \mathbf{j})| = |3\mathbf{i} - 5\mathbf{j} - \mathbf{i} - \mathbf{j})| = |2\mathbf{i} - 6\mathbf{j}| = \sqrt{2^2 + (-6)^2} = \sqrt{4 + 36} = \sqrt{40} = 2\sqrt{10}.$$

Below is a geometric illustration of the vectors from Problem 4 (b).



Problem 5. Find the vector in $\mathbb{R}^3$ that has the same direction as $\langle 6, 2, -3 \rangle$, but has length 4.

Let $\mathbf{a} = \langle 6, 2, -3 \rangle$. Recall that the unit vector (has length 1) $\mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|}$ has the same direction as the vector $\mathbf{a}$. Note that for any scalar c,

$$|c\mathbf{u}| = |c|\,|\mathbf{u}| = |c|.$$

Thus, if $c = 4$, the direction of $4\mathbf{u}$ is the same as the direction of $\mathbf{u}$, and hence $\mathbf{a}$ (multiplying a vector by a positive scalar does not change its direction), and its magnitude is 4. We can explicitly find what $4\mathbf{u}$ is! We have

$$4\mathbf{u} = \frac{4}{|\mathbf{a}|}\mathbf{a} = \frac{4}{\sqrt{6^2 + 2^2 + (-3)^2}}\langle 6, 2, -3 \rangle = \frac{4}{7}\langle 6, 2, -3 \rangle = \langle \frac{24}{7}, \frac{8}{7}, -\frac{12}{7} \rangle.$$