Section 11.4: The Comparison Tests

The Direct Comparison Test Suppose that Σa_n and Σb_n are series such that
$$b_n \geqslant a_n \geqslant 0 \quad \text{for all } n.$$
(i) If Σb_n is convergent then Σa_n is also convergent.
(ii) If Σa_n is divergent then Σb_n is also divergent.

One series we tend to use to compare other series (not always, though) is the p-series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where p is any real number. ***You will be expected to remember the following crucial fact about the convergence and the divergence of p-series.***

1 The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

Another series that can be used for comparison is the geometric series, $\sum_{n=1}^{\infty} ar^{n-1}$, where $a \neq 0$. Remember that the geometric series converges to $\frac{a}{1-r}$ if $|r| < 1$ and it diverges if $|r| \geq 1$.

Problem 1. Use the Direct Comparison Test to determine whether the following series are convergent or divergent. **You do not have to determine the sum if the series is convergent.**

(a) $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{\sqrt{n^3+4n+3}}$ (b) $\sum_{k=1}^{\infty} \frac{k\sin^2(k)}{1+k^3}$ (**Hint:** $\sin^2(k) \leq 1$).

(c) $\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n-2}$ (**Hint:** Compare to a geometric series.) (d) $\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$

(a) The dominating terms in the numerator and the denominator are $\sqrt[3]{n}$ and $\sqrt{n^3}$, respectively. Note that
$$\frac{\sqrt[3]{n}}{\sqrt{n^3}} = \frac{n^{1/3}}{n^{3/2}} = \frac{1}{n^{3/2-1/3}} = \frac{1}{n^{7/6}},$$
and $\sum_{n=1} \frac{1}{n^{7/6}}$ is a convergent p-series since $7/6 > 1$. Since $\sqrt{n^3+4n+3} \geq \sqrt{n^3}$, then
$$\frac{\sqrt[3]{n}}{\sqrt{n^3+4n+3}} \leq \frac{\sqrt[3]{n}}{\sqrt{n^3}} = \frac{1}{n^{7/6}},$$
for all $n \geq 1$, since making the denominator of a fraction smaller creates a larger fraction. Therefore,
$$\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{\sqrt{n^3+4n+3}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{7/6}}.$$
Since $\sum_{n=1}^{\infty} \frac{1}{n^{7/6}}$ is convergent, then by the Direct Comparison Theorem, $\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n-2}$ **is convergent.**

(b) Applying the hint: Given that $\sin^2(k) \leq 1$, then since $\frac{k}{1+k^3} \geq 0$ for all $k \geq 1$, then

$$\frac{k\,\sin^2(k)}{1+k^3} = \frac{k}{1+k^3} \cdot \sin^2(k) \leq \frac{k}{1+k^3} \cdot 1 = \frac{k}{1+k^3}$$

for all $k \geq 1$. The dominating terms in the numerator and the denominator of $\frac{k}{1+k^3}$ are k and k^3, respectively. Note that

$$\frac{k}{k^3} = \frac{1}{k^2},$$

and $\sum\limits_{k=1} \frac{1}{k^2}$ is a convergent p-series since $2 > 1$. Since $1 + k^3 \geq k^3$, then

$$\frac{k}{1+k^3} \leq \frac{k}{k^3} = \frac{1}{k^2},$$

for all $k \geq 1$, since making the denominator of a fraction smaller creates a larger fraction. Therefore,

$$\sum_{k=1}^{\infty} \frac{k\,\sin^2(k)}{1+k^3} \leq \sum_{k=1}^{\infty} \frac{k}{1+k^3} \leq \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Since $\sum\limits_{k=1}^{\infty} \frac{1}{k^2}$ is convergent, then by the Direct Comparison Theorem, $\boldsymbol{\sum\limits_{k=1}^{\infty} \frac{k\,\sin^2(k)}{1+k^3}}$ **is convergent.**

(c) The dominating terms in the numerator and the denominator are 4^{n+1} and 3^n, respectively. Note that

$$\frac{4^{n+1}}{3^n} = \frac{4^2}{3}\left(\frac{4^{n-1}}{3^{n-1}}\right) = \frac{16}{3}\left(\frac{4}{3}\right)^{n-1},$$

and that $\sum\limits_{n=1}^{\infty} \frac{16}{3}\left(\frac{4}{3}\right)^{n-1}$ is a divergent geometric series since $|4/3| \geq 1$. Since $3^n - 2 \leq 3^n$ for all $n \geq 1$, then

$$\frac{4^{n+1}}{3^n - 2} \geq \frac{4^{n+1}}{3^n} = \frac{16}{3}\left(\frac{4}{3}\right)^{n-1},$$

since making the denominator of a fraction larger creates a smaller fraction. Therefore,

$$\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n - 2} \geq \sum_{n=1}^{\infty} \frac{16}{3}\left(\frac{4}{3}\right)^{n-1}.$$

$\sum\limits_{n=1}^{\infty} \frac{16}{3}\left(\frac{4}{3}\right)^{n-1}$ is divergent, by the Direct Comparison Theorem, $\boldsymbol{\sum\limits_{n=1}^{\infty} \frac{4^{n+1}}{3^n - 2}}$ **is divergent.**

(d) The dominating terms in the numerator and the denominator are $2k \cdot k^2 = 2k^3$ and $k(k^2)^2 = k^5$, respectively. Note that

$$\frac{2k^3}{k^5} = \frac{2}{k^2},$$

and $\sum_{k=1} \frac{2}{k^2} = 2 \sum_{k=1} \frac{1}{k^2}$ is 2 times a convergent p-series since $2 > 1$, which is also convergent. Since $(2k-1)(k^2-1) \le 2k \cdot k^2$ and $(k+1)(k^2+4)^2 \ge k(k^2)^2$, then

$$\frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2} \le \frac{2k^3}{k^5} = \frac{2}{k^2},$$

for all $n \ge 1$, since making the numerator fraction larger and its denominator smaller creates a larger fraction. Therefore,

$$\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2} \le \sum_{k=1} \frac{2}{k^2}.$$

Since $\sum_{k=1} \frac{2}{k^2}$ is convergent, then by the Direct Comparison Theorem, $\sum_{\mathbf{k=1}}^{\infty} \frac{\mathbf{(2k-1)(k^2-1)}}{\mathbf{(k+1)(k^2+4)^2}}$ **is convergent.**

> **The Limit Comparison Test** Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms. If
>
> $$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$
>
> where c is a finite number and $c > 0$, then either both series converge or both diverge.

Problem 2. Use the Limit Comparison Test to determine whether the following series are convergent or divergent AND investigate why the Direct Comparison Test is not useful for each of the series. **You do not have to determine the sum if the series is convergent.**

(a) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ (b) $\sum_{n=1}^{\infty} \frac{5+2n}{(1+n^2)^2}$ (c) $\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right)$

(a) Why can't we apply the Direct Comparison Test for this series? Note that

$$\frac{1}{n\sqrt{n^2-1}} \ge \frac{1}{n\sqrt{n^2}} = \frac{1}{n^2},$$

but the series $\sum_{n=1} \frac{1}{n^2}$ is a convergent p-series since $2 > 1$. That is, $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}} \ge \sum_{n=2}^{\infty} \frac{1}{n^2}$, so the DCT is not useful. Therefore, we apply the Limit Comparison Test.

If $a_n = \frac{1}{n\sqrt{n^2-1}}$ and $b_n = \frac{1}{n^2}$, then

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n\sqrt{n^2-1}} = \lim_{n\to\infty} \frac{n/n}{\sqrt{n^2-1}/n} = \lim_{n\to\infty} \frac{1}{\sqrt{1-1/n^2}} = \frac{1}{1} = 1 > 0$, so $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ converges by the Limit Comparison Test with the convergent series $\sum_{n=2}^{\infty} \frac{1}{n^2}$.

(b) Why can't we apply the Direct Comparison Test for this series? Note that

$$\frac{5+2n}{(1+n^2)^2} \geq \frac{2n}{(1+n^2)^2},$$

BUT $\frac{2n}{(1+n^2)^2} \not\geq \frac{2n}{(n^2)^2} = \frac{2}{n^3}$, so the DCT is not useful. Therefore, we apply the Limit Comparison Test.

Note that the dominating term in the numerator and the denominator are $2n$ and $(n^2)^2 = n^4$, respectively. Also note that $\frac{2n}{n^4} = \frac{2}{n^3}$. Therefore, we compare our given series to the p series $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

Use the Limit Comparison Test with $a_n = \frac{5+2n}{(1+n^2)^2}$ and $b_n = \frac{1}{n^3}$:

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3(5+2n)}{(1+n^2)^2} = \lim_{n\to\infty} \frac{5n^3+2n^4}{(1+n^2)^2} \cdot \frac{1/n^4}{1/(n^2)^2} = \lim_{n\to\infty} \frac{\frac{5}{n}+2}{\left(\frac{1}{n^2}+1\right)^2} = 2 > 0.$ Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent p-series $[p = 3 > 1]$, the series $\sum_{n=1}^{\infty} \frac{5+2n}{(1+n^2)^2}$ also converges.

(c) We can only say that $\sin^2\left(\frac{1}{n}\right) \leq 1$, but the series $\sum_{n=1}^{\infty} 1$ is divergent, so the DCT is not useful in this case. Therefore, we will use the Limit Comparison Theorem.

Let $a_n = \sin^2(1/n)$ and $b_n = 1/n^2$. Then $\sum a_n$ and $\sum b_n$ are series with positive terms and

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin^2(1/n)}{1/n^2} &= \lim_{n\to\infty} \left(\frac{\sin(1/n)}{1/n}\right)^2 \\
&= \left(\lim_{n\to\infty} \frac{\sin(1/n)}{1/n}\right)^2 \\
&\overset{\text{L'Hop}}{=} \left(\lim_{n\to\infty} \frac{\cos(1/n)\cdot(-1/n^2)}{-1/n^2}\right)^2 \\
&= \lim_{n\to\infty} \cos(1/n) \\
&= \cos(0) \\
&= 1.
\end{aligned}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series ($p = 2 > 1$), the series $\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right)$ also converges by the LCT.