## Section 5.4: Indefinite Integrals & the Net Change Theorem

**Problem 1.** Find the general indefinite integrals and evaluate the definite integral.

(a) $\int (2+3^x)\,dx$, (b) $\int \left(\frac{1+r}{r}\right)^2 dr$, (c) $\int_0^{1/\sqrt{3}} \frac{t^2-1}{t^4-1}\,dt$.

(a) $\int (2+3^x)\,dx = 2x + \frac{3^x}{\ln(3)} + C$

* RECALL THAT: $\frac{d}{dx} b^x = \ln(b)\cdot b^x$, $b>0$, $b\neq 1$.

(b) $\int \left(\frac{1+r}{r}\right)^2 dr = \int \frac{(1+r)^2}{r^2}\,dr = \int \frac{1+2r+r^2}{r^2}\,dr$

$= \int \left(\frac{1}{r^2} + \frac{2r}{r^2} + \frac{r^2}{r^2}\right) dr$

$= \int (r^{-2} + 2r^{-1} + 1)\,dr = -r^{-1} + 2\ln(|r|) + r + C$

(c) $\int_0^{1/\sqrt{3}} \frac{t^2-1}{t^4-1}\,dt = \int_0^{1/\sqrt{3}} \frac{t^2-1}{(t^2)^2-1^2}\,dt = \int_0^{1/\sqrt{3}} \frac{\cancel{t^2-1}}{\cancel{(t^2-1)}(t^2+1)}\,dt = \int_0^{1/\sqrt{3}} \frac{1}{t^2+1}\,dt$

$= \arctan(t)\Big]_0^{1/\sqrt{3}}$

$= \arctan\left(\frac{1}{\sqrt{3}}\right) - \arctan(0)$

$= \frac{\pi}{6} - 0 = \frac{\pi}{6}$

* RECALL: $\arctan(y) = x \Updownarrow y = \tan(x)$

so since $\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{1}{\sqrt{3}} = \frac{1/2}{\sqrt{3}/2}$ when $x = \frac{\pi}{6}$, $\arctan(1/\sqrt{3}) = \frac{\pi}{6}$

Similarly since $\tan(x) = 0$ when $x = 0$ (since $\sin(0)=0$, $\cos(0)=1$), $\arctan(0) = 0$.

**Problem 2.** The velocity function in $(m/s)$ for a particle moving along a line is given by

$$v(t) = t^2 - 2t - 3.$$

(a) Find the displacement of the particle during the time interval $[1,4]$.
(b) Find the distance traveled by the particle during the time interval $[1,4]$.

(a) The displacement of the particle during the time interval $[1,4]$ is

$$\int_1^4 v(t)\,dt = \int_1^4 (t^2-2t-3)\,dt = \frac{t^3}{3} - t^2 - 3t \Big|_1^4 = F(4) - F(1)$$

Let $F(t) = \frac{t^3}{3} - t^2 - 3t$

Then $F(4) = \frac{4^3}{3} - 4^2 - 3\cdot 4 = \frac{-20}{3}$

$F(1) = -\frac{11}{3}$.

$$= -\frac{20}{3} - \left(-\frac{11}{3}\right)$$

$$= -\frac{9}{3} = -3$$

(b) The distance traveled by the particle during the time interval $[1,4]$ is

$$\int_1^4 |v(t)|\,dt.$$

Note that $t^2 - 2t - 3 = (t+1)(t-3) = 0$ when $t = -1$ or $t = 3$. Then $v(t)$ changes sign at $t = 3$, and since $v(t) = (t+1)(t-3) \le 0$ when $1 \le t \le 3$ and $v(t) \ge 0$ when $3 \le t \le 4$, then

$$\int_1^4 |v(t)|\,dt = \int_1^3 |v(t)|\,dt + \int_3^4 |v(t)|\,dt$$

$$= -\int_1^3 v(t)\,dt + \int_3^4 v(t)\,dt$$

$$= -(F(3) - F(1)) + (F(4) - F(3)), \text{ where } F(t) = \frac{t^3}{3} - t^2 - 3t.$$

$$= -\left(-9 - \left(-\frac{11}{3}\right)\right) + \left(-\frac{20}{3} - (-9)\right)$$

$$= 9 - \frac{11}{3} - \frac{20}{3} + 9 = \frac{23}{3}.$$

$F(4) = \frac{4^3}{3} - 4^2 - 3\cdot 4$

$= \frac{-20}{3}$.

$F(3) = \frac{3^3}{3} - 3^2 - 3^2$

$= -9$

$F(1) = \frac{1}{3} - 1 - 3 = \frac{-11}{3}$

**Problem 3.** A bacterial population is 4000 at time $t = 0$ and its rate of growth is $1000 \cdot 2^t$ bacteria per hour after $t$ hours. What is the population after one hour?

Let $P(t)$ = population of bacteria after $t$ hours. We are given that $P(0) = 4000$ and that $P'(t) = 1000 \cdot 2^t$.

↑ rate of growth of bacteria.

By the Net Change Theorem,

$$P(1) - P(0) = \int_0^1 P'(t)\,dt,$$ or equivalently,

$$P(1) = P(0) + \int_0^1 P'(t)\,dt$$

$$= 4000 + \int_0^1 1000 \cdot 2^t\,dt = 4000 + 1000 \int_0^1 2^t\,dt = 4000 + 1000\left[\frac{2^1 - 2^0}{\ln(2)}\right] = 4000 + \frac{1000}{\ln(2)} \approx 5443 \ \frac{\text{bac.}}{\text{hr.}}$$