Section 12.3: The Dot Product

Problem 1. Let $\mathbf{a} = \langle 1, 2, 3 \rangle$, $\mathbf{b} = \langle 0, 1, 3 \rangle$, and $\mathbf{c} = \langle 2, -1, -1 \rangle$. Find the following dot products.
(a) $\mathbf{a} \cdot \mathbf{b}$, (b) $\mathbf{a} \cdot \mathbf{c}$, (c) $\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$, (d) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$,
(e) $(2\mathbf{a}) \cdot \mathbf{b}$, (f) $2(\mathbf{a} \cdot \mathbf{b})$, (g) $\mathbf{a} \cdot (2\mathbf{b})$.

(a) $\vec{a} \cdot \vec{b} = \langle 1,2,3 \rangle \cdot \langle 0,1,3 \rangle = 1 \cdot 0 + 2 \cdot 1 + 3 \cdot 3 = 0 + 2 + 9 = 11$

(b) $\vec{a} \cdot \vec{c} = \langle 1,2,3 \rangle \cdot \langle 2,-1,-1 \rangle = 1 \cdot 2 + 2(-1) + 3(-1) = 2 - 2 - 3 = -3$

(c) By (a) and (b), $\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} = 11 - 3 = 8$.

(d) $\vec{a} \cdot (\vec{b} + \vec{c}) = \langle 1,2,3 \rangle \cdot \left(\langle 0,1,3 \rangle + \langle 2,-1,-1 \rangle \right) = \langle 1,2,3 \rangle \cdot \langle 2,0,2 \rangle$
$= 1 \cdot 2 + 2 \cdot 0 + 3 \cdot 2 = 8$

(e) $(2\vec{a}) \cdot \vec{b} = \left(2 \langle 1,2,3 \rangle \right) \cdot \langle 0,1,3 \rangle = \langle 2,4,6 \rangle \cdot \langle 0,1,3 \rangle = 2 \cdot 0 + 4 \cdot 1 + 6 \cdot 3 = 22$

(f) $2(\vec{a} \cdot \vec{b}) = 2 \cdot 11 = 22$
by (a)

(g) $\vec{a} \cdot (2\vec{b}) = \langle 1,2,3 \rangle \cdot \left(2 \langle 0,1,3 \rangle \right)$
$= \langle 1,2,3 \rangle \cdot \langle 0,2,6 \rangle$
$= 1 \cdot 0 + 2 \cdot 2 + 3 \cdot 6 = 22$

Problem 2. Show that $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is perpendicular (orthogonal) to $5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

Let $\vec{a} = 2\vec{i} + 2\vec{j} - \vec{k} = \langle 2,2,-1 \rangle$ and $\vec{b} = 5\vec{i} - 4\vec{j} + 2\vec{k}$
$= \langle 5,-4,2 \rangle$.

Then

$$\vec{a} \cdot \vec{b} = \langle 2,2,-1 \rangle \cdot \langle 5,-4,2 \rangle$$

$$= 2 \cdot 5 + 2(-4) - 1 \cdot 2 = 10 - 8 - 2 = 0.$$

Therefore $\vec{a}$ and $\vec{b}$ are orthogonal

Problem 3. Let $\mathbf{a} = \langle 4, 7, -4 \rangle$ and $\mathbf{b} = \langle 3, -1, 1 \rangle$. Find the scalar projection and the vector projection of $\mathbf{b}$ onto $\mathbf{a}$.

Recall that

$$\text{comp}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \frac{\langle 4,7,-4 \rangle \cdot \langle 3,-1,1 \rangle}{\sqrt{4^2 + 7^2 + (-4)^2}}$$

$$= \frac{4 \cdot 3 + 7(-1) - 4 \cdot 1}{\sqrt{16 + 49 + 16}} = \frac{12 - 7 - 4}{\sqrt{81}} = \frac{1}{9}.$$

Then

$$\text{proj}_{\vec{a}} \vec{b} = \text{comp}_{\vec{a}} \vec{b} \cdot \frac{a}{|\vec{a}|} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}\right) \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \vec{a}$$

$$= \frac{1}{9^2} \langle 4, 7, -4 \rangle = \left\langle \frac{4}{81}, \frac{7}{81}, \frac{-4}{81} \right\rangle$$

Problem 4. Prove the famous Cauchy-Schwarz Inequality: For any vectors $\mathbf{a}$ and $\mathbf{b}$,

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}| |\mathbf{b}|.$$

Recall that $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos(\theta)$.

Then

$$|\vec{a} \cdot \vec{b}| = \left| |\vec{a}| |\vec{b}| \cos(\theta) \right| = \left| |\vec{a}| |\vec{b}| \right| |\cos(\theta)|$$

$$= |\vec{a}| |\vec{b}| |\cos(\theta)|.$$

Since $-1 \leq \cos(\theta) \leq 1$ for all θ, then $|\cos(\theta)| \leq 1$. Thus,

$$|\vec{a} \cdot \vec{b}| = |\vec{a}| \cdot |\vec{b}| \cdot |\cos(\theta)| \leq |\vec{a}| \cdot |\vec{b}|.$$